Zero-sets of Clifford Analytic Functions with Real Coefficients

Yan Yang and Tao Qian

School of Mathematics and Computational Science, Sun Yat-Sen University, and Department of Mathematics, Faculty of Science and Technology, University of Macau
E-mail: mathyy@sina.com, fsttq@umac.mo

Abstract : In this note we prove that the zero set of any Clifford analytic function f with real coefficients is the disjoint union of real isolated zeroes and the spherical conjugate ones. What is more, we present a technique for computing the zeroes. We also find the preimages $f^{-1}(A)$ for any paravector A.

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1. Introduction

There has been an ample amount of literature discussing zeroes of functions in quaternions and octonions. Niven in [3, 4] first studied zeroes of quaternionic polynomials which further led to the article by Eilenberg and Niven [5] where a fundamental theorem for quaternionic polynomials was established. In [6], they proved that any quaternionic polynomial of degree $n \geq 1$ has at least one zero and there should be two types of zeroes: They are either isolated or spherical ones. In [7], the authors extended the results in [6] to any quaternionic and octonionic analytic functions with real coefficients using geometrical method. In [8], roots of polynomials with bicomplex coefficients are studied. To the authors knowledge, in the higher dimensional cases, there are not so many deep results. In [12], we first studied the zero-sets of polynomials in higher dimensional cases under the structure of Clifford algebra and then extended the results in [6].

In this article, we study the zeroes of Clifford analytic functions with real coefficients. Using a technical method, we introduce a one-to-one correspondence between such a function and a complex function and then extend the results in [7]. We also find the preimages $f^{-1}(A)$ for any paravector A.

We first give some basic knowledge in relation to Clifford algebra ([1,2]). Let $\mathbf{e}_1, ..., \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$; and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \cdots, m$. Let

$$\mathbf{R}^m = \{\underline{x} = x_1\mathbf{e}_1 + \cdots + x_m\mathbf{e}_m : x_j \in \mathbf{R}, j = 1, 2, \cdots, m\}$$

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be identical with the usual Euclidean space $\mathbf{R}^m$, and

$$\mathbf{R}_1^m = \{x = x_0\mathbf{e}_0 + \underline{x} : x_0 \in \mathbf{R}, \underline{x} \in \mathbf{R}^m\}, \text{ where } \mathbf{e}_0 = 1.$$

An element in $\mathbf{R}_1^m$ is called a *paravector*. For $x \in \mathbf{R}_1^m$, it consists of a scalar part and a vector part. We use the dotations

$$x_0 = \mathrm{Sc}(x), \ \underline{x} = \mathrm{Vec}(x).$$

The real (or complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m$, denoted by $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), is the associative algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_m$ over the real (or complex) field $\mathbf{R}$ (or $\mathbf{C}$). A general element in $\mathbf{R}^{(m)}$ (or $\mathbf{C}^{(m)}$), therefore, is of the form $x = \sum_S x_S\mathbf{e}_S$, where $\mathbf{e}_S = \mathbf{e}_{i_1}\mathbf{e}_{i_2}\cdots\mathbf{e}_{i_l}, x_S \in \mathbf{R}(\text{or } \mathbf{C})$, and S runs over all the ordered subsets of $\{1, 2, \cdots, m\}$, namely

$$S = \{1 \le i_1 < i_2 < \cdots < i_l \le m\}, \quad 1 \le l \le m.$$

We define the conjugation of $\mathbf{e}_S$ to be $\overline{\mathbf{e}}_S = \overline{\mathbf{e}}_{i_l}\cdots\overline{\mathbf{e}}_{i_1}, \overline{\mathbf{e}}_j = -\mathbf{e}_j$. This induces the Clifford conjugate $\overline{x} = x_0 - \underline{x}$ of a paravector $x = x_0 + \underline{x}$.
The product between x and y in $\mathbf{R}_1^m$, denoted by xy is split into three parts: a scalar part, a vector part and a bivector part, that is

$$xy = (x_0y_0 + \underline{x}\cdot\underline{y}) + (x_0\underline{y} + y_o\underline{x}) + \underline{x}\wedge\underline{y},$$

where

$$\underline{x}\cdot\underline{y} = -\sum_{i=1}^m x_iy_i,$$

$$\underline{x}\wedge\underline{y} = \sum_{i=1}^m\sum_{j=i+1}^m (x_iy_j - x_jy_i)\mathbf{e}_i\mathbf{e}_j.$$

In particular,

$$xx = {x_0}^2 - \sum_{i=1}^m {x_i}^2 + 2x_0\underline{x} = 2x_0x - |x|^2,$$

where

$$|x|^2 = x\overline{x} = \sum_{i=0}^m {x_i}^2.$$

It is easy to see that $|x^n| = |x|^n$.

In the following, the so-called Clifford-Heaviside functions

$$P^{\pm}(\underline{x}) = \frac{1}{2}(1 \pm \mathbf{i}\frac{\underline{x}}{|\underline{x}|})$$

will play an important role, which were first introduced by Sommen in [9] and McIntosh in [10]. Introducing spherical coordinates in $\mathbf{R}^m$, we have $\underline{x} = r\underline{\omega}, r = |\underline{x}| \in [0, \infty), \ \underline{\omega} \in S^{m-1}$, where S^{m-1} is the unit sphere in $\mathbf{R}^m$. Thus,

$$P^{\pm}(\underline{\omega}) = \frac{1}{2}(1 \pm \mathbf{i}\underline{\omega}).$$

They are self adjoint mutually orthogonal primitive idempotents:

$$P^+(\underline{\omega}) + P^-(\underline{\omega}) = 1,\ P^+(\underline{\omega})P^-(\underline{\omega}) = P^-(\underline{\omega})P^+(\underline{\omega}) = 0,\ (P^\pm(\underline{\omega}))^2 = P^\pm(\underline{\omega}).$$

Furthermore, we have

$$P^\pm(\underline{\omega})\underline{\omega} = \underline{\omega}P^\pm(\underline{\omega}) = \mp\mathbf{i}P^\pm(\underline{\omega}).$$

The properties of $P^\pm(\underline{\omega})$ are discussed in [11].

2. Zero-sets of Clifford analytic functions with real coefficients

In this section, we will consider the following Clifford analytic function with paravector variable $x \in \mathbf{R}_1^m$ and real coefficients,

$$f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n/x^n,$$

where $a_n, b_n \in \mathbf{R}$.

Definition 2.1 *If $f(z)$ has a Laurent expansion with real coefficients in $r < |z| < R$, that is*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n/z^n,$$

then $f(x)$ is defined as

$$f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n/x^n,$$

where $x \in \mathbf{R}_1^m$ and $r < |x| < R$. If $f(x)$ can be written as this form, that we call it Clifford analytic function.

Note From the norm estimation for $|x^n|$ for Clifford paravectors the above definition is justified.

In [12], we have known that if $x = x_0 + \underline{x} \in \mathbf{R}_1^m$, then

$$x^n = A_n(x)x + B_n(x), n = 1, 2, \cdots$$

where A_n and B_n are real-valued functions of x defined by the recurrent formulas:

$$\begin{aligned} A_{n+1}(x) &= 2\mathrm{Sc}(x)A_n(x) - |x|^2 A_{n-1}(x) \\ B_{n+1}(x) &= -|x|^2 A_n(x), \end{aligned}$$

where

$$\begin{aligned} A_1(x) &= 1 \\ A_2(x) &= 2\mathrm{Sc}(x) \\ B_1(x) &= 0 \\ B_2(x) &= -|x|^2. \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n[A_n(x)x + B_n(x)] + \sum_{n=1}^{\infty} \frac{b_n[A_n(x)x + B_n(x)]}{|x|^{2n}} \\ &= \left[\sum_{n=0}^{\infty} a_n A_n(x) + \sum_{n=1}^{\infty} \frac{b_n A_n(x)}{|x|^{2n}}\right] x + \left[\sum_{n=0}^{\infty} a_n B_n(x) + \sum_{n=1}^{\infty} \frac{b_n B_n(x)}{|x|^{2n}}\right] \\ &= A(x)x + B(x), \end{aligned}$$

denoting $A_0(x) = 0, B_0(x) = 1$.

Note As we have known in [12], given any $x \in \mathbf{R}_1^m$, $A_i(x)$ and $B_i(x)$ depend not on x but on its scalar part x_0 and the modulus of its vector part $|\underline{x}|$. Thus, we have

Lemma 2.1[12] *If two paravectors $x = x_0 + \underline{x},\ y = y_0 + \underline{y}$ with $x_0 = y_0, |\underline{x}| = |\underline{y}|$, then $A_i(x) = A_i(y),\ B_i(x) = B_i(y)$ and hence $A(x) = A(y),\ B(x) = B(y)$.*

Definition 2.2[12] *If $w_1 = \alpha + \mathrm{Vec}(w_1)$ and $w_2 = \alpha + \mathrm{Vec}(w_2)$ are two different paravectors with $|\mathrm{Vec}(w_1)| = |\mathrm{Vec}(w_2)|$, then they are said to be spherical conjugate to each other.*

Proposition 2.1 *Assume that $w_1 = \alpha + \mathrm{Vec}(w_1)$ is a zero of $f(x)$, then any paravector that is spherical conjugate to w_1 is also a zero of it.*

Proof If $f(w_1) = 0$, then we have

$$\begin{aligned} f(w_1) &= \sum_{n=0}^{\infty} a_n[A_n(w_1)w_1 + B_n(w_1)] + \sum_{n=1}^{\infty} \frac{b_n[A_n(w_1)w_1 + B_n(w_1)]}{|w_1|^{2n}} \\ &= A(w_1)w_1 + B(w_1) = 0, \end{aligned}$$

thus $A(w_1) = B(w_1) = 0$.

For any $w = \alpha + \mathrm{Vec}(w)$ with $|\mathrm{Vec}(w)| = |\mathrm{Vec}(w_1)|$, using Lemma 2,1, we have $A(w) = A(w_1), B(w) = B(w_1)$.

Therefore, $f(w) = A(w)w + B(w) = A(w_1)w + B(w_1) = 0$. This completes the proof.

Definition 2.3[12] *Given $f(x)$, then any of its zeroes generating a family of zeroes that are spherical conjugate to each other is called a spherical zero. A zero that is not spherical is called an isolated zero.*

From Proposition 2,1, we know that

Corollary 2.1 *$f(x)$ has no isolated non-real zeroes.*

Next, we will introduce a technique to solve the equation $f(x) = 0$.

Firstly, we need a Lemma.

Lemma 2.2 *If $f(z)$ has a Laurent expansion with real coefficients in $r < |z| < R$, that* *is*

$$f(z)=\sum_{n=0}^{\infty}a_nz^n+\sum_{m=1}^{\infty}b_m/z^m,$$

when $r<|x|<R$, *we have*

(1)$P^+(\underline{\omega})f(x)=f(x)P^+(\underline{\omega})=f(x_0-\mathbf{i}|x|)P^+(\underline{\omega})$
(2)$P^-(\underline{\omega})f(x)=f(x)P^-(\underline{\omega})=f(x_0+\mathbf{i}|x|)P^-(\underline{\omega})$
(3)$f(x)=f(x_0-\mathbf{i}|x|)P^+(\underline{\omega})+f(x_0+\mathbf{i}|x|)P^-(\underline{\omega})$

Proof (1) Using the properties of $P^+(\underline{\omega})$, we have

$$\begin{aligned}
f(x)P^+(\underline{\omega}) &= f(x_0+|\underline{x}|\underline{\omega})P^+(\underline{\omega})\\
&= \left[\sum_{n=0}^{\infty}a_nx^n+\sum_{n=1}^{\infty}b_n/x^n\right]P^+(\underline{\omega})\\
&= \sum_{n=0}^{\infty}a_nx^nP^+(\underline{\omega})+\sum_{n=1}^{\infty}\frac{b_n\overline{x}^nP^+(\underline{\omega})}{|x|^{2n}}\\
&= \sum_{n=0}^{\infty}a_n[xP^+(\underline{\omega})]^n+\sum_{n=1}^{\infty}\frac{b_n[\overline{x}P^+(\underline{\omega})]^n}{|x|^{2n}}\\
&= \sum_{n=0}^{\infty}a_n(x_0-\mathbf{i}|\underline{x}|)^nP^+(\underline{\omega})+\sum_{n=1}^{\infty}\frac{b_n(x_0+\mathbf{i}|\underline{x}|)^nP^+(\underline{\omega})}{|x|^{2n}}\\
&= \left[\sum_{n=0}^{\infty}a_n(x_0-\mathbf{i}|\underline{x}|)^n+\sum_{n=1}^{\infty}b_n/(x_0-\mathbf{i}|\underline{x}|)^n\right]P^+(\underline{\omega})\\
&= f(x_0-\mathbf{i}|\underline{x}|)P^+(\underline{\omega}).
\end{aligned}$$

(2) Similar to (1).
(3)

$$\begin{aligned}
f(x) &= f(x)[P^+(\underline{\omega})+P^-(\underline{\omega})]\\
&= f(x)P^+(\underline{\omega})+f(x)P^-(\underline{\omega})]\\
&= f(x_0-\mathbf{i}|x|)P^+(\underline{\omega})+f(x_0+\mathbf{i}|x|)P^-(\underline{\omega}).
\end{aligned}$$

This completes the proof.

For $f(x)=\sum_{n=0}^{\infty}a_nx^n+\sum_{n=1}^{\infty}b_n/x^n$, where $r<|x|<R$. Using Lemma 2.2, we have

$$\begin{aligned}
f(x)=0 &\iff f(x_0-\mathbf{i}|\underline{x}|)P^+(\underline{\omega})+f(x_0+\mathbf{i}|\underline{x}|)P^-(\underline{\omega})=0\\
&\iff f(x_0-\mathbf{i}|\underline{x}|)P^+(\underline{\omega})=0 \text{ and } f(x_0+\mathbf{i}|\underline{x}|)P^-(\underline{\omega})=0\\
&\iff f(x_0-\mathbf{i}|\underline{x}|)=0 \text{ and } f(x_0+\mathbf{i}|\underline{x}|)=0\\
&\iff f(z)=0.
\end{aligned}$$

Note Note that $f(z)=0$ is an equation of real coefficients. It, therefore, has complex conjugate roots.

Corollary 2.2 *If $\alpha \pm \mathbf{i}\beta, \beta > 0$ are solutions of $f(z) = 0$, then $\alpha + \beta\underline{\omega}$ is a spherical zero of $f(x)$.*

From the above discussion, we can obtain the conclusion as follows:

Theorem 2.1 *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n/x^n, r < |x| < R$ be any Clifford analytic functions with real coefficients, then it has two types of zeroes. The zeroes are either isolated real roots or spherical zeroes. What is more, there exists a one-to-one correspondence between its real isolated zeroes and the real roots of $f(z)$, as well as a one-to-one correspondence between the spherical zeroes of $Q_n(x)$ and the pairs of complex conjugate zeroes of $f(z)$.*

In particular, for polynomial $Q_n(x) = \sum_{j=0}^{n} a_n x^n$ with real coefficients, we have

Corollary 2.3 *The zero-set of $Q_n(x)$ is*

$$S = \{\alpha_1 + \beta_1\underline{\omega}, \cdots, \alpha_s + \beta_s\underline{\omega}, \gamma_1, \cdots, \gamma_t\}$$

if

$$S = \{\alpha_1 \pm \mathbf{i}\beta_1, \cdots, \alpha_s \pm \mathbf{i}\beta_s, \gamma_1, \cdots, \gamma_t,\ \alpha_j, \beta_j, \gamma_k \text{ are reals }, \beta_j > 0, j = 1, \cdots, s, k = 1, \cdots, t\}$$

is the zero-set of $Q_n(z)$. The multiplicity of the zero of $Q_n(x)$ is the same as that of $Q_n(z)$.

Corollary 2.4 *If $Q_n(x)$ has spherical zeroes $\alpha_1 + \beta_1\underline{\omega}, \cdots, \alpha_s + \beta_s\underline{\omega}$ with multiplicity $j_1, \cdots, j_s$ and isolated real roots $\gamma_1, \cdots, \gamma_t$ with multiplicity $k_1, \cdots, k_t$, then*

$$Q_n(x) = a_n[x^2 - 2\alpha_1 + (\alpha_1^2 + \beta_1^2)]^{j_1} \cdots [x^2 - 2\alpha_s + (\alpha_s^2 + \beta_s^2)]^{j_s}(x - \gamma_1)^{k_1} \cdots (x - \gamma_t)^{k_t}.$$

On the other hand, if $Q_n(x)$ can be written as above, then the zero-set of it is

$$S = \{\alpha_1 + \beta_1\underline{\omega}, \cdots, \alpha_s + \beta_s\underline{\omega}, \gamma_1, \cdots, \gamma_t\}.$$

3. The root-set of $f(x) = A$

In this section, we will consider the roots of $f(x) = A$, where $A \in \mathbf{R}_1^m$ and $A \notin \mathbf{R}$. We note that $f(x) = A$ has only isolated non-real roots. In fact, $f(\alpha) \in \mathbf{R}$, $f(\alpha) \neq A$ if $\alpha \in \mathbf{R}$ and if $f(w) = A$, then $f(\overline{w}) = \overline{A} \neq A$.

Next, we will find the roots of it. For $f(x) = A, A = A_0 + |\underline{A}|\underline{\omega_0}$, we have

$$\begin{aligned}
f(x_0 + y\underline{\omega_0}) = A \iff & f(x_0 + y\underline{\omega_0})[P^+(\underline{\omega_0}) + P^-(\underline{\omega_0})] = A[P^+(\underline{\omega_0}) + P^-(\underline{\omega_0})] \\
\iff & f(x_0 - \mathbf{i}y)P^+(\underline{\omega_0}) + f(x_0 + \mathbf{i}y)P^-(\underline{\omega_0}) \\
& = (A_0 - \mathbf{i}|\underline{A}|)P^+(\underline{\omega_0}) + (A_0 + \mathbf{i}|\underline{A}|)P^-(\underline{\omega_0}) \\
\iff & f(x_0 - \mathbf{i}y)P^+(\underline{\omega_0}) = (A_0 - \mathbf{i}|\underline{A}|)P^+(\underline{\omega_0}) \text{ and} \\
& f(x_0 + \mathbf{i}y)P^-(\underline{\omega_0}) = (A_0 + \mathbf{i}|\underline{A}|)P^-(\underline{\omega_0}) \\
\iff & f(x_0 - \mathbf{i}y) = A_0 - \mathbf{i}|\underline{A}| \text{ and } f(x_0 + \mathbf{i}y) = A_0 + \mathbf{i}|\underline{A}| \\
\iff & f(z) = A_0 + \mathbf{i}|\underline{A}|.
\end{aligned}$$

From the discussion above, we can obtain

Theorem 3.1 *The root-set of* $f(x) = A, A = A_0 + |\underline{A}|\underline{\omega_0}$ *is*

$$S = \{\alpha + \beta\underline{\omega_0} : \text{ if } \alpha + \mathbf{i}\beta \text{ is a root of } f(z) = A_0 + \mathbf{i}|\underline{A}|\}.$$

The multiplicity of $\alpha + \beta\underline{\omega_0}$ *is the same as that of* $\alpha + \mathbf{i}\beta$ *as a root of* $f(z) = A_0 + \mathbf{i}|\underline{A}|$.

In particular, we have

Corollary 3.2 *Let* $A = A_0 + |\underline{A}|\underline{\omega_0}$ *be a non zero element. For* $m \in N - \{0\}$*, the polynomial* $P(x) = x^m - A$ *has:*
(1) m *distinct non-real isolated zeroes* $\alpha_1 + \beta_1\underline{\omega_0}, \cdots, \alpha_m + \beta_m\underline{\omega_0}$*, if* A *is non-real number.*
(2) s *spherical zeroes, and an isolated real zero, if* A *is a real number and* $m = 2s + 1$.
(3) $s - 1$ *spherical zeroes, and two distinct isolated real zeroes, if* A *is a positive real number and* $m = 2s$.
(4) s *spherical zeroes, if* A *is a negative real number and* $m = 2s$.

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