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that when $s(t)=\rho (t)e^{iP (t)},$ where $A(t)$ is real-valued, band-limited with minimal bandwidth B and $P (t)$ is real-valued, as the restriction on the real line of some entire function, then s is an analytic signal if and only if $P(t)$ is a linear function, and with $P (t)=a_0+a_1t$ there holds $a_1\leq B.$ In the case s is a mono-component. This generalizes the corresponding result obtained by Xia and Cohen in 1999 in which $P (t)$ is assumed to be a real-valued polynomial.

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Dear Editor

We send our paper "An Application of Entire Function Theory to Analytic Signals" submitting the Journal of analysis and application".

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Sincerely Yours

Guantie DENG and Tao QIAN

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*Manuscript

An Application of Entire Function Theory to Analytic Signals

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Abstract

Analytic signals of finite energy in signal analysis are identical with non-tangential boundary limits of functions in the related Hardy spaces. With this identification this paper studies a subclass of the analytic signals that, with the amplitude-phase representation $s(t) = \rho(t)e^{iP(t)}, \rho(t) \geq 0$, satisfy the relation $P'(t) \geq 0$ a.e. Signals in this subclass are called mono-components, and, in the case, the phase derivative $P'(t)$ is called analytic instantaneous frequency of s. This paper proves that when $s(t) = \rho(t)e^{iP(t)}$, where $A(t)$ is real-valued, band-limited with minimal bandwidth B and $P(t)$ is real-valued, as the restriction on the real line of some entire function, then s is an analytic signal if and only if $P(t)$ is a linear function, and with $P(t) = a_0 + a_1 t$ there holds $a_1 \leq B$. In the case s is a mono-component. This generalizes the corresponding result obtained by Xia and Cohen in 1999 in which $P(t)$ is assumed to be a real-valued polynomial.

Key Words Analytic Signal, Hardy Space, Phase, Mono-component, Entire Function

1 Introduction

There have been unfaded interests in the concept analytic signals since Gabor first introduced it in 1946 ([3], [9]). It is a natural one since it deals with signals, or functions, whose Fourier spectra are supported in the right half of the real line, or, in other words, those possessing only non negative Fourier frequencies ([5]). They, in such a way, correspond to the so called *physically realizable signals.* A close study of this concept is related to the so called analytic instantaneous frequency. If s is a real-valued signal, a candidate of instantaneous frequency of s is the phase derivative $\phi'(t)$, where the phase function $\phi(t)$ is defined through

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$s(t) + iHs(t) = \rho(t)e^{i\phi(t)}, \rho \geq 0$, a.e., and Hs is the Hilbert transform of s. In order to have a qualified frequency function, it would be right to require $\phi'(t) \geq 0$ a.e. Cohen pointed out ([1]) that non-negativity of Fourier frequency does not necessarily imply nonnegativity of analytic phase derivative. In other words, not every analytic signal has an analytic instantaneous frequency function. It is necessary to single out a subclass of analytic signals that have a.e. non-negative analytic phase derivative. The class of mono-components is the subclass of analytic signals that have nonnegative analytic phase derivatives, defined as analytic instantaneous frequency functions. A large pool of mono-components have been identified ([6], [7]), including boundary values of Blaschke products of finite and infinite zeros, of singular inner functions, of starlike and p-starlike functions, and of the basic functions in the Takenaka-Malmquist system, etc. As generalizations of the complex monomials $z^n, |z| < 1$, and the trigonometric functions $e^{itz}, t > 0, \mathrm{z} > 0$, they have been used in adaptive mono-component decompositions of functions ([8]).

Writing a signal in its amplitude-phase representation $s(t) = A(t)(t)e^{iP(t)}$, by saying that s is a chirp signal, it means that $P(t)$ is a polynomial of some degree larger than 1. The result of [9] amounts to say that chirp signals of band-limited amplitude cannot be analytic signals. If $P(t)$ is restricted to be a polynomial, then the only chance for s to be an analytic signal is that $P(t)$ is of degree 1, and, in the case, s has to be a mono-component. The present paper generalizes the result by Xia and Cohen to entire functions $P(z)$ with real values on the real line. We provide an alternative proof that is sophisticated and, more general.

2 Main Result

By the *essential compact support of* $a(\omega) \in L^p, 1 \leq p < \infty$, we mean the smallest compact set S such that

$$\int |a(\omega)|^p d\omega = \int_S |a(\omega)|^p d\omega.$$

We note that if $A(x)$ is real-valued of finite energy, then its Fourier transform, denoted by $a(\omega)$, enjoys the conjugate-symmetric property, viz. $a(-\omega) = \overline{a(\omega)}$. This implies that if $a(\omega)$ has an essential compact support, then the essential compact support is symmetric with respect to the origin. We recall the Paley-Wiener Theorem asserting that an L^2-function $A(x)$ is extendable to become an entire function with exponential type B, ie.

$$|A(z)| \leq Ce^{By}, \qquad C, B > 0, \quad z = x + iy, \tag{1}$$

if and only if $\mathrm{esssupp}\hat{A} \subset [-B, B]$, or, equivalently, $A(x)$ has bandwidth B. Let $A(x)$ be real-valued with bandwidth B. If, moreover, the essential compact support of its Fourier transform $a(\omega)$ is precisely contained in $[-B, B]$, viz.

$$\int_{-B}^{-B+\delta} |a(\omega)| d\omega > 0 \;\text{ and }\; \int_{B-\delta}^{B} |a(\omega)| d\omega > 0$$

for all small enough $\delta > 0$, then the Paley-Wiener Theorem implies that $A(z)$ satisfies the sharp estimate (1) (the constant B cannot be made smaller). In the case $[-B, B]$ is called the *minimum bandwidth* of $A(x)$.

Our main result is as follows.

Theorem 1.1 Assume that $A(x)$ is real-valued, band-limited, and of finite energy,

$$A(x) = \int_{-B}^{B} a(\omega)e^{ix\omega}d\omega, \tag{2}$$

where B is the minimum bandwidth of $A(x)$, and $P(z)$ is an entire function of z,

$$P(z) = \sum_{0}^{\infty} a_n z^n, \tag{3}$$

with real coefficients $a_n \in \mathbb{R}$, $n = 0, 1, 2, \cdots$. Then the fact that $A(x)e^{iP(x)}$ is the boundary value of a function in H^p $(0 < p < \infty)$ of the upper-half complex plane implies that $P(z)$ must be of degree 1, i.e., $P(z) = a_0 + a_1 z$. Furthermore, $a_1 \geq B$.

Proof

The relation (2) shows that $A(x)$ is the restriction to the real line of the entire function given by

$$A(z) = \int_{-B}^{B} a(\omega)e^{iz\omega}d\omega, \qquad z = x + iy, \tag{4}$$

of exponential type B. Therefore, $A(z)e^{iP(z)}$ is an entire function. Since, by assumption, $A(x)e^{iP(x)}$ is the boundary value of a function in the H^p of the upper-half plane, the uniqueness of analytic function implies that the Hardy space function coincides with $A(z)e^{iP(z)}$, $\text{Im}z > 0$.

The inequality

$$\int_{-\infty}^{+\infty} \frac{2\log^+ |A(x)|dx}{\pi(1+x^2)} \leq \log \int_{-\infty}^{+\infty} \frac{|A(x)|^2 dx}{\pi(1+x^2)},$$

implies that $A(z)$ belongs to the class C :

$$\{A(z) \ : \ A(z) \text{ is of exponential type and } \int_{-\infty}^{+\infty} \log^+ |A(x)|/(1+x^2)dx < \infty\}$$

(see [4], p.115).

Therefore, $A(z)$ satisfies the relation

$$\log |A(z)| = By + o(|z|), \ y = \text{Re}z \geq 0, \tag{5}$$

everywhere outside a system of exceptional disks of finite view ([4], p.116). Note that at this point we use the fact that B is the minimum bandwidth. As consequence, for any $\varepsilon > 0, \theta \in (0, \pi)$, there exists a sequence $\{R_n\}$, independent of θ, such that $R_n \to \infty$ as $n \to \infty$, and

$$\log |A(R_n e^{i\theta})| \geq BR_n \sin\theta - \varepsilon R_n. \tag{6}$$

Since $f(t) = A(t)e^{iP(t)}$ is the boundary value of a function $f(z) = A(z)e^{iP(z)}$ in H^p of the upper-half plane, the function $|f(z)|^p$ is subharmonic function. Hence, for $z = x + iy, y > 0$,

$$|f(z)|^p \leq \int_{|\zeta - z| < y} |f(\zeta)|^p \frac{d\lambda(\zeta)}{\pi y^2} \leq \frac{1}{\pi y^2} \int_0^{2y} \int_{\mathbb{R}} |f(\xi + i\eta)|^p d\xi \, d\eta \leq \frac{2||f||_p^p}{\pi y}.$$

where

$$||f||_p^p = \sup\{\int_{\mathbb{R}} |f(\xi + i\eta)|^p d\xi : \ \eta > 0\}.$$

So, there is a positive constant $C > 0$ such that

$$|f(x + iy)|^p = |A(x + iy)|^p \exp(-p\mathrm{Im}P(x + iy)) \leq \frac{C}{y}, \quad y > 0.$$

Taking logarithm on the last obtained inequality and replacing y by $R_n \sin\theta, \theta \in (0, \pi)$, by invoking (6), we have

$$pBR_n \sin\theta - \varepsilon p R_n - p\mathrm{Im}\ P(R_n e^{i\theta}) \leq -\log R_n + \log C - \log \sin\theta. \tag{7}$$

Let $u(z) = -\mathrm{Im}\ P(z)$, $u^+(z) = \max\{u(z), 0\}$ and $u^-(z) = \max\{-u(z), 0\}$. Since $P(z)$ is an entire function of z with Taylor expansion (2), where $a_n \in \mathbb{R}$, $n = 0, 1, 2, \cdots$ are real coefficients, we have

$$u(Re^{i\theta}) = \sum_{k=0}^{\infty} (-a_k R^k) \sin k\theta. \tag{8}$$

This implies that

$$-a_k R^k = \frac{2}{\pi} \int_0^{\pi} u(Re^{i\theta}) \sin k\theta d\theta, \quad k = 0, 1, 2, ...$$

In particular, for $k = 1, R > 0$,

$$-a_1 = \frac{2}{\pi R} \int_0^{\pi} u(Re^{i\theta}) \sin\theta d\theta = \frac{2}{\pi R} \int_0^{\pi} u^+(Re^{i\theta}) \sin\theta d\theta - \frac{2}{\pi R} \int_0^{\pi} u^-(Re^{i\theta}) \sin\theta d\theta.$$

Therefore,

$$\frac{2}{\pi R} \int_0^{\pi} |u(R_n e^{i\theta})| \sin\theta d\theta = a_1 + \frac{4}{\pi R} \int_0^{\pi} u^+(R_n e^{i\theta}) \sin\theta d\theta.$$

By (7), there is a positive constant C_2 such that

$$\frac{2}{\pi R_n}\int_0^{\pi} |u(R_ne^{i\theta})|\sin\theta d\theta \le |a_1| + \frac{4}{\pi R_n}\int_0^{\pi} u^+(R_ne^{i\theta})\sin\theta d\theta \le C_2. \tag{9}$$

For $k \ge 2$,

$$|a_k| \le \frac{2}{\pi R_n^k}\int_0^{\pi} |u(R_ne^{i\theta})\sin k\theta| d\theta \le \frac{2}{\pi R_n^k}\int_0^{\pi} |u(R_ne^{i\theta})| k\sin\theta d\theta \le \frac{kC_2}{R_n^{k-1}}.$$

So $a_k = 0$ for $k \ge 2$. $P(z)$ must be of degree 1, i.e., $P(z) = a_0 + a_1 z$. Furthermore, by (7), for any $\theta \in (0, \pi)$,

$$pBR_n\sin\theta - \varepsilon pR_n - p(a_0 + a_1R_n\sin\theta) \le -\log R_n + \log C - \log\sin\theta.$$

By taking $\theta = \frac{\pi}{2}$, we have $a_1 \ge B - \varepsilon$. Since $\varepsilon > 0$ is arbitrary small, we arrive at $a_1 \ge B$.

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