

Phase Derivative of Monogenic Signals in Higher Dimensional Spaces

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Abstract : In the Clifford algebra setting of a Euclidean space on the boundary of a domain it is natural to define a monogenic (analytic) signal to be the boundary value of a monogenic (analytic) function inside the domain. The question is how to define the corresponding phase and phase derivative. In this paper we give an answer to these questions in the unit ball and in the upper-half space. Among the possible candidates of phases and phase derivatives we decided that the right ones are those that give rise to, as in the one dimensional signal case, the equal relations between the mean of the Fourier frequency and the mean of the phase derivative, and the positivity of the phase derivative of the shifted Cauchy kernel.

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1 Introduction

In time-frequency analysis the concept analytic signals is introduced (see [Co], [Pi]). It is well-known that for a real-valued square integrable signal f in the whole time range, the function $f + \mathbf{iH}f$, where \mathbf{H} is the Hilbert transformation on the line, is the boundary value of an analytic function in the upper-half complex plane. The function $f + \mathbf{iH}f$ is called the *analytic signal associated with f* , that is, in fact, the boundary value of a constant multiple of the Cauchy integral of f in the upper-half complex plane. In the periodic signals case, for a given function $f \in L^2(T)$, written in the form

$$f(e^{it}) = \sum_{k=-\infty}^{\infty} c_k e^{ikt}, \quad t \in [0, 2\pi],$$

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where

$$c_k = \frac{1}{2\pi} \int_0^{2\pi} f(e^{it}) e^{-ikt} dt,$$

the *circular Hilbert transform* $\tilde{\mathbf{H}}$ is given by

$$\tilde{\mathbf{H}}[f](e^{it}) = -\mathbf{i} \sum_{k=-\infty}^{\infty} \text{sgn}(k) c_k e^{ikt}, \quad (1.1)$$

where $\text{sgn}(k) = 1$, if $k > 0$; 0 , if $k = 0$; and -1 , if $k < 0$.

The integral form of the circular Hilbert transform is given by

$$\tilde{\mathbf{H}}(f)(e^{it}) = \frac{1}{2\pi} \lim_{\varepsilon \rightarrow 0^+} \int_{|t-x|>0} \cot\left(\frac{t-x}{2}\right) f(e^{ix}) dx.$$

The function

$$f + \mathbf{i}\tilde{\mathbf{H}}f = c_0 + 2 \sum_{k=1}^{\infty} c_k e^{ikt}$$

is called the *analytic signal associated with f* . It is the boundary value of an analytic function inside the unit disc. Unlike the upper-half plane case, it is not a constant multiple of the boundary value of the Cauchy integral of f . So the circular Hilbert transform does not correspond to the imagine part of the Cauchy integral of f . In fact, it corresponds to the imagine part of the Schwarz integral of f , that is the conjugate Poisson integral. In the upper-half plan case the Cauchy kernel and the Schwarz kernel coincide. In general, Hilbert transformation on the boundary of a domain is defined to be the operator that maps the scalar parts of the boundary values of analytic functions in the domain to their non-scalar parts ([Be]). This concept is extended to higher dimensional spaces in the Clifford algebra setting ([QY]).

From the definition of circular Hilbert transform, it is easy to see that $\tilde{\mathbf{H}}(c_0) = 0$, $\tilde{\mathbf{H}}[\cos(kt)] = \sin(kt)$, $\tilde{\mathbf{H}}[\sin(kt)] = -\cos(kt)$.

In the polar coordinate representation we have

$$f + \mathbf{i}\tilde{\mathbf{H}}f = A(t)[\cos \theta(t) + \mathbf{i} \sin \theta(t)] = A(t)e^{\mathbf{i}\theta(t)},$$

where $A(t) = \sqrt{f^2 + (\tilde{\mathbf{H}}f)^2}$ is called *instantaneous amplitude*, and $\theta(t) = \arctg \frac{\tilde{\mathbf{H}}f}{f}$ is called the *instantaneous phase*. The derivative of the phase, $\theta'(t)$, is usually defined to be *instantaneous frequency*.

In order to make the above defined “instantaneous frequency” a qualified concept complying with its physical meaning many researchers require the additional condition $\theta'(t) \geq 0$ a.e. ([QCL], [Q]). In the present paper this point is not insisted.

In higher dimensional spaces, G. Sommer and M. Felsberg et al ([FS], [Fe] and its references thereby) using the Hilbert transform in higher dimensions (Riesz transforms)

first studied the monogenic signals in the upper half spaces and gave the definitions of local amplitude and local phase, which have significant applications in image processing. In addition, Bülow and Sommer have published work on hypercomplex Fourier transforms applied to images, such as [BS1], [BS2] [Bu].

The main purpose of this paper is given the definitions of instantaneous frequency both in the unit ball and in the upper half plane from theory. In our definition instantaneous frequency is real-valued, and the treatment is universal. The proposed definitions have roots in signal analysis for it gives rise to the necessary equal relations between the mean of the Fourier frequency and the mean of the phase derivative. Our definitions also implies the positivity of the phase derivative of the shifted Cauchy kernels.

2 Preliminary

Most of the basic knowledge and notation in relation to Clifford algebra hereby are referred to [BDS] and [DSS].

Let $\mathbf{e}_1, \dots, \mathbf{e}_m$ be *basic elements* satisfying $\mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$, and $\delta_{ij} = 0$ otherwise, $i, j = 1, 2, \dots, m$. Let

$$\mathbf{R}_1^m = \{x_0 + \underline{x}, \underline{x} \in \mathbf{R}^m\},$$

where

$$\mathbf{R}^m = \{\underline{x} = x_1 \mathbf{e}_1 + \dots + x_m \mathbf{e}_m : x_j \in \mathbf{R}, j = 1, 2, \dots, m\}$$

be identical with the usual Euclidean space \mathbf{R}^m .

An element in \mathbf{R}^m is called a *vector*. The real (complex) Clifford algebra generated by $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_m$, denoted by \mathbf{R}_m (\mathbf{C}_m), is the associative algebra over the real (complex) field \mathbf{R} (\mathbf{C}). A general element in \mathbf{R}_m , therefore, is of the form $x = \sum_S x_S \mathbf{e}_S$ and $\mathbf{e}_S = \mathbf{e}_{i_1} \mathbf{e}_{i_2} \dots \mathbf{e}_{i_l}$, and S runs over all the ordered subsets of $\{1, 2, \dots, m\}$, namely

$$S = \{1 \leq i_1 < i_2 < \dots < i_l \leq m\}, \quad 1 \leq l \leq m.$$

The multiplication of two vectors $\underline{x} = \sum_{j=1}^m x_j \mathbf{e}_j$ and $\underline{y} = \sum_{j=1}^m y_j \mathbf{e}_j$ is given by

$$\underline{x}\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$$

with

$$\begin{aligned} \underline{x} \cdot \underline{y} &= - \sum_{j=1}^m x_j y_j = \frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x}) = - \langle \underline{x}, \underline{y} \rangle \\ \underline{x} \wedge \underline{y} &= \sum_{i < j} e_{ij} (x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x}) \end{aligned}$$

being a scalar and a bi-vector, respectively. We denote by $\text{Sc}(\underline{x}y)$ and $\text{Bi}(\underline{x}y)$, respectively. In particular, we have $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2 = -\sum_{j=1}^m x_j^2$.

We define the conjugation and reversion of \mathbf{e}_S are $\bar{\mathbf{e}}_S = \bar{\mathbf{e}}_{i_1} \cdots \bar{\mathbf{e}}_{i_l}$, $\bar{\mathbf{e}}_j = -\mathbf{e}_j$ and $\tilde{\mathbf{e}}_S = \mathbf{e}_{i_1} \cdots \mathbf{e}_{i_l}$. So the Clifford conjugate of a vector \underline{x} is $\bar{\underline{x}} = -\underline{x}$. The Clifford reversion of a vector \underline{x} is $\tilde{\underline{x}} = \underline{x}$. It is easy to verify that $0 \neq \underline{x} \in \mathbf{R}^m$ implies

$$\underline{x}^{-1} = \frac{\bar{\underline{x}}}{|\underline{x}|^2}.$$

The *open ball with center 0 and radius R* in \mathbf{R}^m is denoted by $B(0, R)$ and the unit sphere in \mathbf{R}^m is denoted by S^{m-1} .

The natural inner product between x and y in \mathbf{C}_m , denoted by $\langle x, y \rangle$, is the complex number $\sum_S x_S \bar{y}_S$, where $x = \sum_S x_S \mathbf{e}_S$ and $y = \sum_S y_S \mathbf{e}_S$. The norm associated with this inner product is

$$|x| = \langle x, x \rangle^{\frac{1}{2}} = \left(\sum_S |x_S|^2 \right)^{\frac{1}{2}}.$$

Below we will study functions defined in \mathbf{R}^m taking values in \mathbf{C}_m . So, they are of the form $f(\underline{x}) = \sum_S f_S(\underline{x}) \mathbf{e}_S$, where f_S are complex-valued functions. We will use the *Dirac operator* \underline{D} , where $\underline{D} = \frac{\partial}{\partial x_1} \mathbf{e}_1 + \cdots + \frac{\partial}{\partial x_m} \mathbf{e}_m$. We define the “left” and “right” roles of the operators \underline{D} by

$$\underline{D}f = \sum_{i=1}^m \sum_S \frac{\partial f_S}{\partial x_i} \mathbf{e}_i \mathbf{e}_S$$

and

$$f\underline{D} = \sum_{i=1}^m \sum_S \frac{\partial f_S}{\partial x_i} \mathbf{e}_S \mathbf{e}_i.$$

If $\underline{D}f = 0$ in a domain (open and connected) Ω , then we say that f is *left-monogenic* in Ω ; and, if $f\underline{D} = 0$ in Ω , we say that f is *right-monogenic* in Ω . If f is both left- and right-monogenic, then we say that f is *monogenic*.

We call

$$E(\underline{x}) = \frac{\bar{\underline{x}}}{|\underline{x}|^m}$$

the *Cauchy kernel* in \mathbf{R}^m . It is easy to see that $E(\underline{x})$ is a monogenic function in $\mathbf{R}^m \setminus \{0\}$.

For $\underline{x} = |\underline{x}|\underline{\xi} = r\underline{\xi}$, there is the polar form of the Dirac operator

$$\underline{D} = \underline{\xi} \partial_r - \frac{1}{r} \partial_{\underline{\xi}} = \frac{1}{r} \underline{\xi} (r \partial_r + \bar{\underline{\xi}} \partial_{\underline{\xi}}) = \frac{1}{r} \underline{\xi} (r \partial_r + \Gamma_{\underline{\xi}}),$$

where $\Gamma_{\underline{\xi}}$ is the bi-vector-valued spherical Dirac operator

$$\Gamma_{\underline{\xi}} = \bar{\underline{\xi}} \partial_{\underline{\xi}} = -\sum_{i < j} \mathbf{e}_i \mathbf{e}_j (x_j \partial_{x_j} - x_j \partial_{x_i}).$$

Denoting $\tilde{\Gamma}_{\underline{\xi}} = (m-1)I - \Gamma_{\underline{\xi}}$, where I is the identity operator, then $\tilde{\Gamma}_{\underline{\xi}} - I = (m-2)I - \Gamma_{\underline{\xi}}$.

Remark 2.1: When $m = 2$, let $\underline{\xi} = \cos t \mathbf{e}_1 + \sin t \mathbf{e}_2$, then $\bar{\underline{\xi}} = -\cos t \mathbf{e}_1 - \sin t \mathbf{e}_2$.

$$\partial_{\underline{\xi}} = [-\sin t \mathbf{e}_1 + \cos t \mathbf{e}_2] \frac{\partial}{\partial t}.$$

Therefore,

$$\begin{aligned} \Gamma_{\underline{\xi}} &= \bar{\underline{\xi}} \partial_{\underline{\xi}} \\ &= [-\cos t \mathbf{e}_1 - \sin t \mathbf{e}_2] [-\sin t \mathbf{e}_1 + \cos t \mathbf{e}_2] \frac{\partial}{\partial t} \\ &= \mathbf{e}_2 \mathbf{e}_1 \frac{\partial}{\partial t} \\ &= \bar{\mathbf{e}}_1 \mathbf{e}_2 \frac{\partial}{\partial t}. \end{aligned}$$

In the complex plane, Let $\bar{\mathbf{e}}_1 \mathbf{e}_2 = \mathbf{i}$, then $\underline{\xi} = \cos t \mathbf{e}_1 + \sin t \mathbf{e}_2 = \mathbf{e}_1(\cos t + \sin t \bar{\mathbf{e}}_1 \mathbf{e}_2)$ is isomorphic to $\cos t + \mathbf{i} \sin t$, and $-\Gamma_{\underline{\xi}} = \frac{1}{\mathbf{i}} \frac{\partial}{\partial t}$.

Remark 2.2: When $m = 2$,

$$\tilde{\Gamma}_{\underline{\xi}} - I = -\Gamma_{\underline{\xi}}.$$

3 Fourier expansion and spherical Hilbert transforms

It is well-known that if $f \in L^2(S^{m-1})$, then we have the Fourier-Laplace expansion

$$f(\underline{\xi}) = \sum_{k=0}^{\infty} S_k(f)(\underline{\xi}), \quad \underline{\xi} \in S^{m-1},$$

where $S_k(f) \in \mathcal{H}_k$ are k -spherical harmonics given by

$$S_k(f)(\underline{\xi}) = \dim(\mathcal{H}_k) \frac{1}{\omega_{m-1}} \int_{S^{m-1}} P_{k,m}(\langle \underline{\xi}, \underline{y} \rangle) f(\underline{y}) dS_{\underline{y}},$$

where $P_{k,m}(t)$ are Legendre polynomials of degree k in dimension m , and dS the surface area element (see [SW]).

Thanks to Clifford analysis, if we denote

$$P_k(f)(\underline{\xi}) = \frac{m+k-2-\Gamma_{\underline{\xi}}}{2k+m-2} [S_k(f)(\underline{\xi})]$$

and

$$Q_{k-1}(f)(\underline{\xi}) = \frac{k+\Gamma_{\underline{\xi}}}{2k+m-2} [S_k(f)(\underline{\xi})], \quad k \geq 1,$$

then

$$f(\underline{\xi}) = \sum_{k=0}^{\infty} [P_k(f)(\underline{\xi}) + Q_{k-1}(f)(\underline{\xi})], \quad (3.1)$$

where $P_0(f) = \text{constant}$ and $Q_{-1} = \{0\}$,

$$P_k(f)(\underline{\xi}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} C_{m,k}^+(\underline{\xi}, \underline{y}) f(\underline{y}) dS_{\underline{y}},$$

and

$$Q_{k-1}(f)(\underline{\xi}) = \frac{1}{\omega_{m-1}} \int_{S^{m-1}} C_{m,k-1}^-(\underline{\xi}, \underline{y}) f(\underline{y}) dS_{\underline{y}},$$

where

$$\begin{aligned} C_{m,k}^+(\underline{\xi}, \underline{y}) &= \frac{1}{2-m} [-(m+k-2)C_k^{(m-2)/2}(\langle \underline{\xi}, \underline{y} \rangle) \\ &+ (2-m)C_{k-1}^{(m/2)}(\langle \underline{\xi}, \underline{y} \rangle)(\underline{\xi} \wedge \underline{y})], \end{aligned}$$

and

$$\begin{aligned} C_{m,k-1}^-(\underline{\xi}, \underline{y}) &= \frac{1}{m-2} [kC_k^{(m-2)/2}(\langle \underline{\xi}, \underline{y} \rangle) \\ &+ (2-m)C_{k-1}^{(m/2)}(\langle \underline{\xi}, \underline{y} \rangle)(\underline{\xi} \wedge \underline{y})], \quad k \geq 1, \end{aligned}$$

C_k^v is the Gegenbauer polynomial of degree k associated with v . We call (3.1) the *Fourier expansion* of $f \in L^2(S^{m-1})$.

The component $P_k(f)(\underline{\xi})$ is called an *inner spherical monogenics* of degree k , which is the restriction to the unit sphere of the k -homogeneous left-monogenic function $P_k(f)(r\underline{\xi})$ in the unit ball. The component $Q_{k-1}(f)(\underline{\xi})$ is called an *outer spherical monogenics* of degree $k-1$, which is the restriction to the unit sphere of the $-(m+k-2)$ -homogeneous left-monogenic function $Q_{k-1}(f)(r\underline{\xi})$ in $\mathbf{R}^m \setminus \{0\}$. Therefore, $P_k(f)(\underline{\xi}) \in H_2^+(S^{m-1})$ and $Q_{k-1}(f)(\underline{\xi}) \in H_2^-(S^{m-1})$, where the Hardy spaces $H_2^+(S^{m-1})$ and $H_2^-(S^{m-1})$ are Hilbert subspaces of the L^2 on the sphere under the inner product (6.1), and they are orthogonal complements to each other.

There hold the following relations (see [DSS]):

$$\begin{aligned} -\Gamma_{\underline{\xi}}[P_k(\underline{\xi})] &= kP_k(\underline{\xi}), \\ -\Gamma_{\underline{\xi}}[Q_{k-1}(\underline{\xi})] &= -(k+m-2)Q_{k-1}(\underline{\xi}), \\ (\tilde{\Gamma}_{\underline{\xi}} - I)[P_k(\underline{\xi})] &= (k+m-2)P_k(\underline{\xi}), \quad \text{and} \\ (\tilde{\Gamma}_{\underline{\xi}} - I)[Q_{k-1}(\underline{\xi})] &= -kQ_{k-1}(\underline{\xi}). \end{aligned}$$

For the boundary value of a left monogenic function in the ball, the non-scalar part of it is defined to be the *inner spherical Hilbert transform* of the scalar part. For the boundary value of a left monogenic function defined outside the unit ball, we similarly define *outer*

spherical Hilbert transform. In this paper, we only deal with the inner spherical Hilbert transform, abbreviated as spherical Hilbert transform. The theory for outer spherical Hilbert transform is similar.

The inner and outer spherical Hilbert transform of the function $f \in L^2(S^{m-1})$ are well studied in [BDDE], [BDE], [BV] and [QY]. The first integral representation of the spherical Hilbert transform is given by [BV]. In [QY], by using the Abelian sum of the inner spherical Hilbert transform of f , we obtain the singular integral representation, as follows.

$$\tilde{H}[f](\underline{\xi}) = \lim_{r \rightarrow 1^-} \int_{S^{m-1}} Q_r(\underline{y}, \underline{\xi}) f(\underline{y}) dS_{\underline{y}},$$

where

$$Q_r(\underline{y}, \underline{\xi}) = \frac{1}{\omega_{m-1}} \left[\frac{2}{|\underline{y} - r\underline{\xi}|^m} - \frac{m-2}{r^{m-1}} \int_0^r \frac{\rho^{m-2}}{|\underline{y} - \rho\underline{\xi}|^m} d\rho \right] r\underline{\xi} \wedge \underline{y}.$$

Denoting

$$I_m = \int_0^r \frac{\rho^{m-2}}{|\underline{y} - \rho\underline{\xi}|^m} d\rho,$$

and $t = \langle \underline{y}, \underline{\xi} \rangle$, we have

$$I_{m+2}(r, t) = \frac{1}{m} \left[(m-1 + t \frac{d}{dt}) I_m - \frac{r^{m-1}}{|1 - 2rt + r^2|^{m/2}} \right],$$

while

$$I_2(r, t) = \frac{1}{\sqrt{1-t^2}} \operatorname{arctg} \frac{r-t}{\sqrt{1-t^2}},$$

$$I_3(r, t) = \frac{rt-1}{(1-t^2)\sqrt{1-2rt+r^2}}.$$

The series form of the spherical Hilbert transform is (see [BD], [QY]):

$$\tilde{H}[f](\underline{\xi}) = \sum_{k=1}^{\infty} \left[\frac{k}{k+m-2} P_k(f)(\underline{\xi}) - Q_{k-1}(f)(\underline{\xi}) \right].$$

From the definition of the spherical Hilbert transform, we have

$$\tilde{H}[P_0] = 0,$$

$$\tilde{H}[P_k(f) + Q_{k-1}(f)] = \frac{k}{k+m-2} P_k(f) - Q_{k-1}(f).$$

Remark 3.1: It is shown in [BDE] that

$$\tilde{H}[f](\underline{\xi}) = \sum_{k=1}^{\infty} \frac{-\Gamma_{\underline{\xi}}[S_k(f)(\underline{\xi})]}{k+m-2}. \quad (3.2)$$

Clearly, since f is real-valued, $\tilde{H}[f](\underline{\xi})$ is bi-vector-valued and has the form

$$\underline{\xi} \sum_{k=1}^{\infty} \frac{\partial_{\underline{\xi}} [S_k(f)(\underline{\xi})]}{k+m-2}.$$

Remark 3.2: From Remark 2.1, in the complex plane, using the isomorphic form with $m = 2$, the right hand side of (3.2) becomes

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{\frac{1}{i} \frac{\partial}{\partial t} [c_k e^{ikt} + c_{-k} e^{-ikt}]}{k} &= \sum_{k=1}^{\infty} (c_k e^{ikt} - c_{-k} e^{-ikt}) \\ &= \mathbf{i} \tilde{H}[f](e^{it}). \end{aligned}$$

That means, up to the imaginary unit \mathbf{i} , the spherical Hilbert transform $\tilde{H}[f]$ on S^1 coincides with the circular Hilbert transform defined in (1.1).

Example 3.3: Consider the spherical Hilbert transform of the spherical harmonics $C_k^{\frac{m-2}{2}}(\langle \underline{\xi}, \underline{y} \rangle)$. Using the formula (3.2), we have

$$\begin{aligned} -\Gamma_{\underline{\xi}} \left[\frac{1}{k+m-2} C_k^{\frac{m-2}{2}}(\langle \underline{\xi}, \underline{y} \rangle) \right] &= -\frac{m-2}{k+m-2} C_k^{\frac{m-2}{2}}(\langle \underline{\xi}, \underline{y} \rangle) \Gamma_{\underline{\xi}}[\langle \underline{\xi}, \underline{y} \rangle] \\ &= -\frac{m-2}{k+m-2} C_k^{\frac{m-2}{2}}(\langle \underline{\xi}, \underline{y} \rangle) \underline{y} \wedge \underline{\xi}, \end{aligned}$$

where we invoke the property $\frac{d}{dt} C_k^v(t) = 2v C_{k-1}^{v+1}(t)$. This result can be found in [DSS]. Therefore,

$$P_k(f)(\underline{\xi}) = \text{Sc}[P_k(f)(\underline{\xi})] + \text{Bi}[P_k(f)(\underline{\xi})] = \text{Sc}[P_k(f)(\underline{\xi})] + \tilde{H}\{\text{Sc}[P_k(f)(\underline{\xi})]\}.$$

4 Monogenic signals and phase derivatives on the unit sphere

Let f be a real-valued square integrable signal on the unit sphere S^{m-1} . From the definition of the spherical Hilbert transform, we have $f(\underline{\xi}) + \tilde{H}[f](\underline{\xi}) \in H_2^+(S^{m-1})$, which is the boundary value of a left-monogenic function inside the unit ball. We call $f(\underline{\xi}) + \tilde{H}[f](\underline{\xi})$ the *monogenic signal* associated with f .

For a monogenic signal $f^+(\underline{\xi}) = f(\underline{\xi}) + \tilde{H}[f](\underline{\xi})$, we can write it in the form:

$$\begin{aligned} f + \tilde{H}[f] &= A(f) \left[\frac{f}{A(f)} + \frac{\tilde{H}[f]}{A(f)} \right] \\ &= A(f) \left[\frac{f}{A(f)} + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \frac{|\tilde{H}[f]|}{A(f)} \right] \end{aligned}$$

$$\begin{aligned}
&= A(f)[\cos \theta(\underline{\xi}) + \frac{H[f]}{|H[f]|} \sin \theta(\underline{\xi})] \\
&= A(f)e^{\frac{H[f]}{|H[f]|} \theta(\underline{\xi})},
\end{aligned}$$

where $A(f) = \sqrt{f^2 + \tilde{H}^2[f]}$ is called the *amplitude*, $\theta(\underline{\xi}) = \arctan \frac{|\tilde{H}[f]|}{f}$ the *phase* that is between 0 and $\frac{\pi}{2}$, $\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \theta(\underline{\xi})$ the *phase vector*, and $e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \theta(\underline{\xi})]}$ the *phase direction*. We define the *directional phase derivative* by $\text{Sc} \left\{ [-\Gamma_{\underline{\xi}} \theta(\underline{\xi})] \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right\}$.

Geometrical explanation of instantaneous phase direction

When $m = 2$, by definition a monogenic signal is of the form $f_0 + f_1 \mathbf{e}_1 \mathbf{e}_2$, the instantaneous phase is $e^{\theta \mathbf{e}_1 \mathbf{e}_2}$, where $\theta = \arctan \frac{f_1}{f_0}$. The product of a vector $\underline{a} = a_1 \mathbf{e}_1 + a_2 \mathbf{e}_2 \in \mathbf{R}^2$ and the instantaneous phase $e^{\theta \mathbf{e}_1 \mathbf{e}_2}$, that is

$$\underline{a} e^{\theta \mathbf{e}_1 \mathbf{e}_2} = e^{-(\theta/2) \mathbf{e}_1 \mathbf{e}_2} \underline{a} e^{(\theta/2) \mathbf{e}_1 \mathbf{e}_2},$$

is the rotation of \underline{a} anticlockwise by the angle θ .

When $m = 3$, a monogenic signal is this form $f_0 + f_1 \mathbf{e}_2 \mathbf{e}_3 + f_2 \mathbf{e}_3 \mathbf{e}_1 + f_3 \mathbf{e}_1 \mathbf{e}_2$. Using the Hodge dual, it can be written as $f_0 - (f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3) \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3$. We denote $f_1 \mathbf{e}_1 + f_2 \mathbf{e}_2 + f_3 \mathbf{e}_3$ by \underline{f} . Then the instantaneous phase direction is $e^{-\theta \frac{\underline{f}}{|\underline{f}|} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3}$, where $\theta = \arctan \frac{|\underline{f}|}{f_0}$. Let \underline{a} be a vector. Then

$$e^{-(\theta/2) \frac{\underline{f}}{|\underline{f}|} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3} \underline{a} e^{(\theta/2) \frac{\underline{f}}{|\underline{f}|} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3}$$

is the rotation of \underline{a} along the axis \underline{f} by the angle θ in the clockwise looking from the arrow-head position of \underline{f} . For details, see [DSS] and [Lo].

$A(f) e^{-\theta \frac{\underline{f}}{|\underline{f}|} \mathbf{e}_1 \mathbf{e}_2 \mathbf{e}_3}$ is a rotation of the real value $A(f)$ along the rotation axis $-\mathbf{i} \sin \phi(\underline{\xi}) + \mathbf{j} \cos \phi(\underline{\xi})$ which is orthogonal to the plane spanned by the real axis and the vector $\mathbf{i} \cos \phi(\underline{\xi}) + \mathbf{j} \sin \phi(\underline{\xi})$ by the angle θ in the clockwise looking from the arrow-head position of the rotation axis.

Remark 4.1: Particularly, for a monogenic signal of this form $f_0 + f_1 \mathbf{e}_2 \mathbf{e}_3 + f_2 \mathbf{e}_3 \mathbf{e}_1$, in [FS], it is written in this form $f_0 + f_1 \mathbf{i} + f_2 \mathbf{j}$. Then

$$\begin{aligned}
f_0 + f_1 \mathbf{i} + f_2 \mathbf{j} &= A(f) \left[\frac{f_0}{A(f)} + \frac{f_1 \mathbf{i} + f_2 \mathbf{j}}{A(f)} \right] \\
&= A(f) \left[\frac{f_0}{A(f)} + \frac{f_1 \mathbf{i} + f_2 \mathbf{j}}{|f_1 \mathbf{i} + f_2 \mathbf{j}|} \frac{|f_1 \mathbf{i} + f_2 \mathbf{j}|}{A(f)} \right] \\
&= A(f) [\cos \theta(\underline{\xi}) + \mathbf{i} \sin \theta(\underline{\xi}) \cos \phi(\underline{\xi}) + \mathbf{j} \sin \theta(\underline{\xi}) \sin \phi(\underline{\xi})] \\
&= A(f) e^{[\mathbf{i} \cos \phi(\underline{\xi}) + \mathbf{j} \sin \phi(\underline{\xi})] \theta(\underline{\xi})},
\end{aligned}$$

where $A(f) = \sqrt{(f_0)^2 + (f_1)^2 + (f_2)^2}$, $\theta(\underline{\xi}) = \arctan \frac{|f_1 \mathbf{i} + f_2 \mathbf{j}|}{f_0} = \arctan \frac{\sqrt{(f_1)^2 + (f_2)^2}}{f_0}$, $\phi(\underline{\xi}) = \arctan \frac{f_2}{f_1}$. In [FS], the geometrical explanation of $A(f) e^{[\mathbf{i} \cos \phi(\underline{\xi}) + \mathbf{j} \sin \phi(\underline{\xi})] \theta(\underline{\xi})}$ is a rotation of

the real value $A(f)$ along the rotation axis $-\mathbf{i} \sin \phi(\underline{\xi}) + \mathbf{j} \cos \phi(\underline{\xi})$ which is orthogonal to the plane spanned by the real axis and the vector $\mathbf{i} \cos \phi(\underline{\xi}) + \mathbf{j} \sin \phi(\underline{\xi})$ by the angle θ in the clockwise looking from the arrow-head position of the rotation axis.

Next, we discuss how to define instantaneous frequency for monogenic signals. First we consider instantaneous frequency for analytic signals on the circle.

For an analytic signal $f^+ = f + \mathbf{i}\tilde{H}f$, we have

$$f^+(e^{it}) = A(t)[\cos \theta(t) + \mathbf{i} \sin \theta(t)] = A(t)e^{i\theta(t)}.$$

In the complex plane, the instantaneous frequency for an analytic signal is defined to be the derivative of the instantaneous phase $\theta'(t)$. There are various ways to compute $\theta'(t)$:

- (a) $\frac{1}{\mathbf{i}} \frac{\partial}{\partial t} [\mathbf{i}\theta(t)];$
- (b) $\frac{1}{\mathbf{i}} \frac{\partial}{\partial t} [\cos \theta(t)] / \mathbf{i} \sin \theta(t);$
- (c) $\frac{1}{\mathbf{i}} \frac{\partial}{\partial t} [\mathbf{i} \sin \theta(t)] / \cos \theta(t);$
- (d) $\frac{1}{\mathbf{i}} \frac{\partial}{\partial t} [e^{i\theta(t)}] / e^{i\theta(t)};$
- (e) $\text{Re} \left[\frac{\frac{1}{\mathbf{i}} \frac{\partial}{\partial t} f^+(e^{it})}{f^+(e^{it})} \right].$

As we know, the operator $\frac{1}{\mathbf{i}} \frac{\partial}{\partial t}$ corresponds to $-\Gamma_{\underline{\xi}} = \tilde{\Gamma}_{\underline{\xi}} - I$ in the dimension $m = 2$. Therefore, in higher dimensions we may have two alternative methods to define phase derivative or instantaneous frequency of a monogenic signal.

Method 1: Define instantaneous frequency by

$$\text{Sc} \left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \right] \left[f^+(\underline{\xi}) \right]^{-1} \right\}. \quad (4.1)$$

We can show that

$$\begin{aligned} & \text{Sc} \left\{ \left[-\Gamma_{\underline{\xi}} f^+(\underline{\xi}) \right] \left[f^+(\underline{\xi}) \right]^{-1} \right\} \\ &= \text{Sc} \left\{ \left[-\Gamma_{\underline{\xi}} e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \theta(\underline{\xi})]} \right] \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \theta(\underline{\xi})]} \right]^{-1} \right\} \\ &= \text{Sc} \left\{ -\Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin \theta(\underline{\xi}) \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \theta(\underline{\xi})]} \right]^{-1} \right\} + \text{Sc} \left\{ \left[-\Gamma_{\underline{\xi}} \theta(\underline{\xi}) \right] \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right\}. \quad (4.2) \end{aligned}$$

In the complex plane, the first part of equation (4.2) reduces to zero and the second part of it is just the phase derivative.

To show (4.2), we have

$$-\Gamma_{\underline{\xi}} f^+(\underline{\xi})$$

$$\begin{aligned}
&= -\Gamma_{\underline{\xi}} \left\{ A(f) \left[\cos \theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin \theta(\underline{\xi}) \right] \right\} \\
&= -\Gamma_{\underline{\xi}} A(f) \left[\cos \theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin \theta(\underline{\xi}) \right] - A(f) \Gamma_{\underline{\xi}} \left[\cos \theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin \theta(\underline{\xi}) \right]. \\
-\Gamma_{\underline{\xi}} \left[\cos \theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin \theta(\underline{\xi}) \right] &= \sin \theta \Gamma_{\underline{\xi}} \theta - \cos \theta \Gamma_{\underline{\xi}} \theta \frac{\tilde{H}[f]}{|\tilde{H}[f]|} - \Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin \theta.
\end{aligned}$$

Then

$$\begin{aligned}
&[-\Gamma_{\underline{\xi}} f^+(\underline{\xi})][f^+(\underline{\xi})]^{-1} \\
&= [-\Gamma_{\underline{\xi}} f^+(\underline{\xi})] \left\{ \frac{1}{A(f)} \left[\cos \theta(\underline{\xi}) - \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin \theta(\underline{\xi}) \right] \right\} \\
&= \frac{-\Gamma_{\underline{\xi}} A(f)}{A(f)} + [-\Gamma_{\underline{\xi}} e^{[\tilde{H}[f]|\tilde{H}[f]|\theta(\underline{\xi})]}] \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} \\
&= \frac{-\Gamma_{\underline{\xi}} A(f)}{A(f)} - \Gamma_{\underline{\xi}} \left[\cos \theta(\underline{\xi}) + \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin \theta(\underline{\xi}) \right] \left[\cos \theta(\underline{\xi}) - \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \sin \theta(\underline{\xi}) \right]
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\text{Sc} \left\{ [-\Gamma_{\underline{\xi}} f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} \right\} \\
&= \text{Sc} \left\{ \left[-\Gamma_{\underline{\xi}} e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right] \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} \right\} \\
&= \text{Sc} \left\{ -\Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin \theta(\underline{\xi}) \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} - \Gamma_{\underline{\xi}} \theta(\underline{\xi}) \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \right\} \\
&= \text{Sc} \left\{ -\Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin \theta(\underline{\xi}) \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} \right\} + \text{Sc} \left\{ [-\Gamma_{\underline{\xi}} \theta(\underline{\xi})] \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right\}.
\end{aligned}$$

Method 2: Define instantaneous frequency by

$$\text{Sc} \left\{ [(\tilde{\Gamma}_{\underline{\xi}} - I) f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} \right\}.$$

Similarly, we can show that

$$\begin{aligned}
&\text{Sc} \left\{ [(\tilde{\Gamma}_{\underline{\xi}} - I) f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} \right\} \\
&= (m-2) + \text{Sc} \left\{ [-\Gamma_{\underline{\xi}} f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} \right\} \\
&= (m-2) + \text{Sc} \left\{ -\Gamma_{\underline{\xi}} \left(\frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right) \sin \theta(\underline{\xi}) \left[e^{[\frac{\tilde{H}[f]}{|\tilde{H}[f]|}\theta(\underline{\xi})]} \right]^{-1} \right\} \\
&\quad + \text{Sc} \left\{ [-\Gamma_{\underline{\xi}} \theta(\underline{\xi})] \frac{\tilde{H}[f]}{|\tilde{H}[f]|} \right\}. \tag{4.3}
\end{aligned}$$

Note that in higher dimensions, phase derivative or instantaneous frequency defined through **Method 1** or **Method 2** are no longer equal to the directional derivative of the instantaneous phase.

Remark 4.2: For $P_k(\underline{\xi})$, by using the two methods, the instantaneous frequencies obtained are, respectively, k and $k + m - 2$. When $m = 2$, P_k reduces to e^{ikt} , and $k + m - 2$ reduces to k .

Remark 4.3: If $f(\underline{x})$ is left-monogenic, then $\underline{D}f = 0$. Using the polar form of the Dirac operator

$$\underline{D} = \frac{1}{r} \underline{\xi} (r \partial_r + \Gamma_{\underline{\xi}}),$$

we have $-\Gamma_{\underline{\xi}} f(\underline{x}) = r \partial_r f(\underline{x})$. Therefore, $-\Gamma_{\underline{\xi}} f(\underline{\xi}) = \lim_{r \rightarrow 1^-} \partial_r f(\underline{x})$. In many occasions it is easier to compute with the operator ∂_r than the operator $\Gamma_{\underline{\xi}}$.

Now we justify the above given definitions of instantaneous frequency. The following relation plays an important role in one dimensional signal analysis: Let s be an analytic signal of finite energy and $s(t) = |s(t)|e^{i\varphi(t)}$. Then in both the classical sense ([Co]) and the extended sense ([DQY], [DQ]),

$$\langle \omega \rangle = \int_0^\infty \omega |\hat{s}(\omega)|^2 d\omega = \int_{-\infty}^\infty \frac{d\varphi(t)}{dt} |s(t)|^2 dt. \quad (4.4)$$

For periodic analytic signals there exists an analogous relation ([DQ])

$$\langle \omega \rangle = \sum_{k=0}^\infty k |c_k|^2 = \int_{-\infty}^\infty \frac{d\varphi(t)}{dt} |s(t)|^2 dt, \quad (4.5)$$

where c_k 's are the Fourier coefficients of f .

For a monogenic signal $f^+(\underline{\xi}) = \sum_{k=0}^\infty P_k(\underline{\xi})$, we have, correspondingly, two methods to define the mean of the Fourier frequency as follows.

Method 1:

$$\langle k \rangle_1 = \sum_{k=0}^\infty k |P_k(\underline{\xi})|^2.$$

In the case we can prove, similarly to (4.5),

$$\langle k \rangle_1 = \int_{S^{m-1}} \text{Sc} \left\{ [-\Gamma_{\underline{\xi}} f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} \right\} |f^+(\underline{\xi})|^2 dS_{\underline{\xi}}.$$

In fact,

$$\begin{aligned} \langle k \rangle_1 &= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \sum_{k=0}^\infty k |P_k(\underline{\xi})|^2 dS_{\underline{\xi}} \\ &= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \sum_{k=0}^\infty k P_k(\underline{\xi}) \overline{P_k(\underline{\xi})} dS_{\underline{\xi}} \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \sum_{k=0}^{\infty} (-\Gamma_{\underline{\xi}}) P_k(\underline{\xi}) \overline{P_k(\underline{\xi})} dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} (-\Gamma_{\underline{\xi}}) f^+(\underline{\xi}) \overline{f^+(\underline{\xi})} dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} [(-\Gamma_{\underline{\xi}}) f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} f^+(\underline{\xi}) \overline{f^+(\underline{\xi})} dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} [(-\Gamma_{\underline{\xi}}) f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} |f^+(\underline{\xi})|^2 dS_{\underline{\xi}} \\
&= \frac{1}{\omega_{m-1}} \int_{S^{m-1}} \text{Sc} \left\{ [(-\Gamma_{\underline{\xi}}) f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} \right\} |f^+(\underline{\xi})|^2 dS_{\underline{\xi}}.
\end{aligned}$$

Method 2:

$$\langle k \rangle_2 = \sum_{k=0}^{\infty} (k + m - 2) |P_k(\underline{\xi})|^2.$$

Using the same technique, we can prove, similarly to (4.5),

$$\langle k \rangle_2 = \int_{S^{m-1}} \text{Sc} \left\{ [(\tilde{\Gamma}_{\underline{\xi}} - I) f^+(\underline{\xi})] [f^+(\underline{\xi})]^{-1} \right\} |f^+(\underline{\xi})|^2 dS_{\underline{\xi}}.$$

For some physical reasons, we need to expand a signal by those of positive analytic instantaneous frequency. This aspect has recently been studied in a series papers of Qian et al ([Q], [QCL], [Q1], [Q2], [QW]). In this complex plane, the study is based on the analytic signals, the boundary values $e^{i\theta_a(t)} = \frac{e^{it} - a}{1 - \bar{a}e^{it}}$, $|a| < 1$, of the corresponding Möbius transforms from the unit disc to the unit disc, called *Fourier atoms*. Observe that ([Ga], [Q])

$$\frac{1}{2\pi} \frac{d\theta_a(t)}{dt} = \frac{1}{2\pi} \frac{1 - |a|^2}{|e^{it} - a|^2}$$

that is the Poisson kernel of the disc at a , and positive, we have non-trivial analytic signals of positive phase derivatives or instantaneous frequencies. Furthermore, $\left\{ \frac{1}{\sqrt{2\pi}} \frac{1 - \bar{a}e^{it}}{|1 - \bar{a}e^{it}|^2} e^{ik\theta_a(t)} \right\}_{k=-\infty}^{\infty}$ is a weighed trigonometric system.

In higher dimensions, it is natural to consider the functions $P_k(\tau_{\underline{a}}(\underline{\xi}))$, where $\tau_{\underline{a}}(\underline{\xi})$ is the Möbius transformation from the unit ball to the unit ball. We first recall Möbius transforms in higher dimensional spaces.

5 Möbius Transforms in higher dimensional spaces

In this section, we will recall well known results of Möbius transformation in \mathbf{R}^m .

Definition 5.1 ^[Ah] *A Möbius transformation is a function $M : \mathbf{R}^m \cup \{\infty\} \longrightarrow \mathbf{R}^m \cup \{\infty\}$, which can be expressed as finite composition of translations, dilations, orthogonal*

transformations and inversions. In fact, each Möbius transformation is a homeomorphism from $\mathbf{R}^m \cup \{\infty\}$ to $\mathbf{R}^m \cup \{\infty\}$.

In \mathbf{R}^m , it is more difficult to describe the Möbius transformation. That is especially because that multiplication in \mathbf{R}^m is not closed. For example, \underline{ab} may be not in \mathbf{R}^m , although \underline{a} and \underline{b} are in \mathbf{R}^m . In order to solve this problem, we need the following Lemma, that is

Lemma 5.1 ^[Ah] *If $a, b, c, d \in \Gamma_m \cup \{0\}$, and satisfy*

(1) $ac^{-1}, d\tilde{c} \in \mathbf{R}^m$, when $c \neq 0$;

(2) $a\tilde{d} - b\tilde{c} = \pm 1$;

(3) $bd^{-1} \in \mathbf{R}^m$, when $c = 0$.

Then $\psi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$ is a Möbius transformation from $\mathbf{R}^m \cup \{\infty\}$ to $\mathbf{R}^m \cup \{\infty\}$ and this expression factorizes to equal $ac^{-1} \pm (c\underline{x}\tilde{c} + d\tilde{c})^{-1}$ when $c \neq 0$ and $\pm a\underline{x}\tilde{a} + bd^{-1}$ when $c = 0$.

This above extension was first worked out by Karl Theodor Vahlen in 1902, and was re-studied by Ahlfors over eighty years later, in 1984.

Note 5.1: The Jacobian of the transform on the sphere is given as ([Ah])

$$J_\psi(\underline{x}) = \left(\frac{|a\tilde{d} - b\tilde{c}|}{|c\underline{x} + d|^2} \right)^m = |c\underline{x} + d|^{-2m}.$$

Example 5.1: $\underline{\omega} = \tau_{\underline{a}}(\underline{x}) = (\underline{x} - \underline{a})(1 - \underline{a}\underline{x})^{-1}$ is a Möbius transformation which maps the unit ball onto itself and $\underline{\omega}^{-1} = \tau_{-\underline{a}}(\underline{x})$.

Unfortunately, the composition of a monogenic function and a Möbius transformation is, in general, no longer monogenic. Möbius transforms themselves, and products of monogenic functions are usually not monogenic functions. In fact, transformations of harmonic and monogenic functions are more rigid here than those in the complex case.

Let $\psi(\underline{x}) = (a\underline{x} + b)(c\underline{x} + d)^{-1}$ be a Möbius transformation in $\mathbf{R}^m \cup \{\infty\}$, then (see [PQ], for instance)

(1) If f is monogenic, then so is

$$\frac{c\underline{x} + d}{|c\underline{x} + d|^m} f(\psi(\underline{x}));$$

and

(2) If h is harmonic, then so is

$$\frac{1}{|c\underline{x} + d|^{m-2}} h(\psi(\underline{x})).$$

Next we consider the frequency of $\frac{1-\widetilde{a\xi}}{|1-\widetilde{a\xi}|^m} P_k \left((\underline{\xi} - \underline{a})(1 - \widetilde{a\xi})^{-1} \right)$.

Example 5.2: $\frac{1-\widetilde{a\xi}}{|1-\widetilde{a\xi}|^m} P_k \left((\underline{\xi} - \underline{a})(1 - \widetilde{a\xi})^{-1} \right)$ is the restriction to the unit sphere of the left-monogenic function $\frac{1-\widetilde{a\underline{x}}}{|1-\widetilde{a\underline{x}}|^m} P_k \left((\underline{x} - \underline{a})(1 - \widetilde{a\underline{x}})^{-1} \right)$ in the unit ball. Using Method 1 and 2 we obtain the instantaneous frequencies $\frac{(m-1)(\langle \underline{a}, \underline{\xi} \rangle - |\underline{a}|^2) + k(1 - |\underline{a}|^2)}{|1-\widetilde{a\xi}|^2}$ and $\frac{(1+k)(1-|\underline{a}|^2) + (m-3)(1-\langle \underline{a}, \underline{\xi} \rangle)}{|1-\widetilde{a\xi}|^2}$ (> 0 when $m > 2$), respectively. In fact, proceeding with the computation given in (4.1), we have

$$\begin{aligned} & -\Gamma_{\underline{\xi}} \left[\frac{1-\widetilde{a\xi}}{|1-\widetilde{a\xi}|^m} P_k \left((\underline{\xi} - \underline{a})(1 - \widetilde{a\xi})^{-1} \right) \right] \\ &= \lim_{r \rightarrow 1^-} r \frac{\partial}{\partial r} \left[\frac{1-\widetilde{a r \underline{\xi}}}{|1-\widetilde{a r \underline{\xi}}|^m} P_k \left((r \underline{\xi} - \underline{a})(1 - \widetilde{a r \underline{\xi}})^{-1} \right) \right] \\ &= \lim_{r \rightarrow 1^-} \frac{-\widetilde{\xi a} |1 - \widetilde{a r \underline{\xi}}|^2 - m(1 - r \widetilde{\xi a})(-\langle \underline{\xi}, \underline{a} \rangle + r |\underline{a}|^2)}{|1 - \widetilde{a r \underline{\xi}}|^{m+2}} P_k \left((r \underline{\xi} - \underline{a})(1 - \widetilde{a r \underline{\xi}})^{-1} \right) \\ &+ \lim_{r \rightarrow 1^-} \frac{1-\widetilde{a r \underline{\xi}}}{|1-\widetilde{a r \underline{\xi}}|^m} r \frac{\partial}{\partial r} \left[P_k \left((r \underline{\xi} - \underline{a})(1 - \widetilde{a r \underline{\xi}})^{-1} \right) \right] \\ &= \frac{-\widetilde{\xi a} |1 - \widetilde{a \underline{\xi}}|^2 - m(1 - \widetilde{\xi a})(-\langle \underline{\xi}, \underline{a} \rangle + |\underline{a}|^2)}{|1 - \widetilde{a \underline{\xi}}|^{m+2}} P_k \left((\underline{\xi} - \underline{a})(1 - \widetilde{a \underline{\xi}})^{-1} \right) \\ &+ \frac{1-\widetilde{a \underline{\xi}}}{|1-\widetilde{a \underline{\xi}}|^m} P_k \left((\underline{\xi} - \underline{a})(1 - \widetilde{a \underline{\xi}})^{-1} \right) k \frac{1 - |\underline{a}|^2}{|1 - \widetilde{a \underline{\xi}}|^2} \\ &= \left[\frac{-\widetilde{\xi a} + (1 - m)|\underline{a}|^2 + m \langle \underline{\xi}, \underline{a} \rangle}{|1 - \widetilde{a \underline{\xi}}|^2} + k \frac{1 - |\underline{a}|^2}{|1 - \widetilde{a \underline{\xi}}|^2} \right] \frac{1-\widetilde{a \underline{\xi}}}{|1-\widetilde{a \underline{\xi}}|^m} P_k \left((\underline{\xi} - \underline{a})(1 - \widetilde{a \underline{\xi}})^{-1} \right) \\ &= Y(\underline{\xi}) \frac{1-\widetilde{a \underline{\xi}}}{|1-\widetilde{a \underline{\xi}}|^m} P_k \left((\underline{\xi} - \underline{a})(1 - \widetilde{a \underline{\xi}})^{-1} \right). \end{aligned}$$

The formula (4.1) implies that the instantaneous frequency is the scalar part of $Y(\underline{\xi})$, that is $\frac{(m-1)(\langle \underline{\xi}, \underline{a} \rangle - |\underline{a}|^2) + k(1 - |\underline{a}|^2)}{|1-\widetilde{a \underline{\xi}}|^2}$. Similarly, adopting Method 2, the instantaneous frequency is $\frac{(m-3)(1-\langle \underline{a}, \underline{\xi} \rangle) + (k+1)(1-|\underline{a}|^2)}{|1-\widetilde{a \underline{\xi}}|^2}$ (> 0 when $m > 2$). The Cauchy kernel $\frac{1-\widetilde{a \underline{\xi}}}{|1-\widetilde{a \underline{\xi}}|^m}$ is the special cases of the above for $k = 0$ and $P_k = 1$. The instantaneous frequency of it is $\frac{(m-1)(\langle \underline{\xi}, \underline{a} \rangle - |\underline{a}|^2)}{|1-\widetilde{a \underline{\xi}}|^2}$ and $\frac{(1-|\underline{a}|^2) + (m-3)(1-\langle \underline{a}, \underline{\xi} \rangle)}{|1-\widetilde{a \underline{\xi}}|^2}$ (> 0 when $m > 2$), respectively.

Conclusion: From the above two examples, we prefer to choose Method 2 to define the instantaneous frequency and the mean of the instantaneous frequency for monogenic signals.

6 A system of $L^2(S^{m-1})$

Consider $L^2(S^{m-1})$, the space of scalar-valued square integrable functions on S^{m-1} , equipped with the inner product

$$(f, g) = \int_{S^{m-1}} \bar{f}(\underline{\xi})g(\underline{\xi})dS_{\underline{\xi}}. \quad (6.1)$$

Next, we will give a monogenic orthogonal system for $L^2(S^{m-1})$.

Theorem 6.1 *Let $\underline{a} \in B(0, 1) \subseteq \mathbf{R}^m$. Denote*

$$\mathcal{F}_{\underline{a}} = \left\{ \frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} Q_l(\tau_{-\underline{a}}(\underline{\xi})), k, l \geq 0 \right\}.$$

Then $\mathcal{F}_{\underline{a}}$ is a complete monogenic orthogonal system for $L^2(S^{m-1})$.

Proof A shorter proof can be given but we provide a more detailed proof for the sake of completeness.

We consider three cases.

Case 1. when $k \neq l$,

$$\left(\frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_l(\tau_{-\underline{a}}(\underline{\xi})) \right) = 0.$$

Case 2. When $k \neq l$,

$$\left(\frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} Q_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} Q_l(\tau_{-\underline{a}}(\underline{\xi})) \right) = 0.$$

Case 3. For any k, l ,

$$\left(\frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} Q_l(\tau_{-\underline{a}}(\underline{\xi})) \right) = 0.$$

To proceed the proof of Case 1, we have

$$\begin{aligned} & \left(\frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), \frac{1 + \widetilde{\underline{a}} \underline{\xi}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_l(\tau_{-\underline{a}}(\underline{\xi})) \right) \\ &= \int_{S^{m-1}} \frac{1 + \underline{\xi} \underline{a}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})) \overline{\frac{1 + \underline{\xi} \underline{a}}{|1 + \underline{\bar{a}} \underline{\xi}|^m} P_l(\tau_{-\underline{a}}(\underline{\xi}))} dS_{\underline{\xi}} \\ &= \int_{S^{m-1}} \frac{1}{|1 + \underline{\bar{a}} \underline{\xi}|^{2m-2}} \bar{P}_k(\tau_{-\underline{a}}(\underline{\xi})) P_l(\tau_{-\underline{a}}(\underline{\xi})) dS_{\underline{\xi}} \end{aligned}$$

$$\begin{aligned}
&= \int_{S^{m-1}} \frac{|1 - \bar{a} \underline{x}|^{2m-2}}{|1 - |\bar{a}|^2|^{2m-2}} \bar{P}_k(\underline{x}) P_l(\underline{x}) \frac{dS_{\underline{x}}}{dS_{\underline{x}}} dS_{\underline{x}} \\
&= \int_{S^{m-1}} \frac{|1 - \bar{a} \underline{x}|^{2m-2}}{|1 - |\bar{a}|^2|^{2m-2}} \bar{P}_k(\underline{x}) P_l(\underline{x}) \frac{|1 - |\bar{a}|^2|^{m-1}}{|1 - \bar{a} \underline{x}|^{2m-2}} dS_{\underline{x}} \\
&= \frac{1}{(1 - |\bar{a}|^2)^{m-1}} \int_{S^{m-1}} \bar{P}_k(\underline{x}) P_l(\underline{x}) dS_{\underline{x}} \\
&= 0.
\end{aligned}$$

Invoking the orthogonality between P_k and Q_l and that between Q_k and $Q_l, k \neq l$, we can prove Case 2 and Case 3.

Next, we will prove completeness of the system. For any $f \in L^2(S^{m-1})$, we have $\left[\frac{1 + \bar{a} \widetilde{\tau_{\underline{a}}(\underline{\xi})}}{|1 + \bar{a} \tau_{\underline{a}}(\underline{\xi})|^m} \right]^{-1} f(\tau_{\underline{a}}(\underline{\xi})) \in L^2(S^{m-1})$, then

$$\left[\frac{1 + \bar{a} \widetilde{\tau_{\underline{a}}(\underline{\xi})}}{|1 + \bar{a} \tau_{\underline{a}}(\underline{\xi})|^m} \right]^{-1} f(\tau_{\underline{a}}(\underline{\xi})) = \sum_{k=0}^{\infty} [P_k(\underline{\xi}) + Q_{k-1}(\underline{\xi})],$$

therefore,

$$f(\underline{\omega}) = \sum_{k=0}^{\infty} \left[\frac{1 + \bar{a} \underline{\omega}}{|1 + \bar{a} \underline{\omega}|^m} P_k(\tau_{-\underline{a}}(\underline{\omega})) + \frac{1 + \bar{a} \underline{\omega}}{|1 + \bar{a} \underline{\omega}|^m} Q_{k-1}(\tau_{-\underline{a}}(\underline{\omega})) \right].$$

We complete the proof.

Remark 6.1: When $\underline{a} = 0$, $\tau_{\underline{a}}(\underline{x}) = \underline{x}$, so \mathcal{F}_0 becomes the Fourier basis of $L^2(S^{m-1})$.

Theorem 6.2 *The system*

$$\mathcal{F}_{\underline{a}} = \left\{ (\sqrt{1 - |\underline{a}|^2})^{m-1} \frac{1 + \bar{a} \underline{\xi}}{|1 + \bar{a} \underline{\xi}|^m} P_k(\tau_{-\underline{a}}(\underline{\xi})), k \geq 0 \right\}$$

is a complete orthonormal basis of $H_2^+(S^{m-1})$ and the instantaneous frequencies of the basis functions are positive.

7 Monogenic signals and phase derivatives in the upper half spaces

If $f \in L^2(\mathbf{R}^m)$, we define the *Fourier transform* of f by

$$\hat{f}(\underline{t}) = \int_{\mathbf{R}^m} e^{-i\langle \underline{x}, \underline{t} \rangle} f(\underline{x}) d\underline{x}$$

and the *inverse Fourier transform* by

$$f(\underline{x}) = \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{i\langle \underline{x}, \underline{t} \rangle} \hat{f}(\underline{t}) d\underline{t}.$$

The Hilbert transform of $f(\underline{x})$ has the following alternative representations

$$\begin{aligned} H[f](\underline{x}) &= \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} \frac{D_{\underline{x}} e^{i\langle \underline{x}, \underline{t} \rangle}}{|\underline{t}|} \hat{f}(\underline{t}) d\underline{t} \\ &= \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} \frac{i\underline{t}}{|\underline{t}|} e^{i\langle \underline{x}, \underline{t} \rangle} \hat{f}(\underline{t}) d\underline{t} \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{|\underline{x}-\underline{t}| > \varepsilon} \frac{\overline{\underline{x}-\underline{t}}}{|\underline{x}-\underline{t}|^{m+1}} f(\underline{t}) d\underline{t} \\ &= -\sum_{j=1}^m R_j(f)(\underline{x}) \mathbf{e}_j, \end{aligned}$$

where $R_j(f)(\underline{x}) = \lim_{\varepsilon \rightarrow 0^+} \int_{|\underline{x}-\underline{t}| > \varepsilon} \frac{x_j - t_j}{|\underline{x}-\underline{t}|^{m+1}} f(\underline{t}) d\underline{t}$ is the j th-Riesz transform of f ([SW]). Clearly, if f is real-valued, then $H[f](\underline{x})$ is vector-valued.

Remark 7.1: It is easily to see that $H^2[f](\underline{x}) = f(\underline{x})$.

Remark 7.2: When $m = 1$, letting $\mathbf{e}_1 = -\mathbf{i}$, then we have $\underline{D} = \frac{1}{\mathbf{i}} \frac{\partial}{\partial x}$. The right hand side of (7.1) becomes

$$\begin{aligned} &\frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{\frac{1}{\mathbf{i}} \frac{\partial}{\partial x} e^{i\langle x, t \rangle}}{|t|} \hat{f}(t) dt \\ &= \frac{1}{(2\pi)} \int_{-\infty}^{\infty} \frac{t}{|t|} e^{i\langle x, t \rangle} \hat{f}(t) dt \\ &= \frac{\mathbf{i}}{(2\pi)} \int_{-\infty}^{\infty} -\text{isgn}(t) e^{i\langle x, t \rangle} \hat{f}(t) dt \\ &= \mathbf{iH}[f](x). \end{aligned}$$

It is well known ([LMQ]) that

$$\begin{aligned} \frac{1}{2}f(\underline{x}) + \frac{1}{2}H[f](\underline{x}) &= \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{i\langle \underline{x}, \underline{t} \rangle} \frac{1}{2} \left(1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|}\right) \hat{f}(\underline{t}) d\underline{t} \\ &= \lim_{x_0 \rightarrow 0^+} \frac{1}{(2\pi)^m} \int_{\mathbf{R}^m} e^{+(x_0 + \underline{x}, \underline{t})} \hat{f}(\underline{t}) d\underline{t} \\ &= f^+(\underline{x}), \end{aligned}$$

where

$$e^{+(x_0 + \underline{x}, \underline{t})} = e^{-x_0|\underline{t}|} e^{i\langle \underline{x}, \underline{t} \rangle} \frac{1}{2} \left(1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|}\right)$$

is left monogenic in \mathbf{R}_1^m ([LMQ]). Therefore, $f(\underline{x}) + H[f](\underline{x}) \in H_2^+(\mathbf{R}^m)$, being the boundary value of a left-monogenic functions in the upper half space \mathbf{R}_1^m .

We call $f(\underline{x}) + H[f](\underline{x})$ the *monogenic signal associated with f* , where f is a real-valued square integral signal on \mathbf{R}^m .

For a monogenic signal $f^+(\underline{x}) = f(\underline{x}) + H[f](\underline{x})$, we write it in the form:

$$\begin{aligned} f + \tilde{H}[f] &= A(f) \left[\frac{f}{A(f)} + \frac{H[f]}{A(f)} \right] \\ &= A(f) \left[\frac{f}{A(f)} + \frac{H[f]}{|H[f]|} \frac{|H[f]|}{A(f)} \right] \\ &= A(f) \left[\cos \theta(\underline{x}) + \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right] \\ &= A(f) e^{[\frac{H[f]}{|H[f]|} \theta(\underline{x})]}, \end{aligned}$$

where $A(f) = \sqrt{f^2 + H^2[f]}$ is called the *amplitude*, $\theta(\underline{x}) = \arctan \frac{|H[f]|}{f}$ the *phase* that is between 0 and $\frac{\pi}{2}$, $\frac{H[f]}{|H[f]|} \theta(\underline{x})$ the *phase vector*, and $e^{[\frac{H[f]}{|H[f]|} \theta(\underline{x})]}$ the *phase direction*. We define the *directional phase derivative* by $\text{Sc} \left\{ [\underline{D}\theta(\underline{x})] \frac{H[f]}{|H[f]|} \right\}$ and define the *phase derivative* or *instantaneous frequency* by

$$\text{Sc} \left\{ [\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1} \right\}.$$

We can prove that

$$\begin{aligned} &\text{Sc} \left\{ [\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1} \right\} \\ &= \text{Sc} \left\{ \left[\underline{D}e^{[\frac{H[f]}{|H[f]|} \theta(\underline{x})]} \right] \left[e^{[\frac{H[f]}{|H[f]|} \theta(\underline{x})]} \right]^{-1} \right\} \\ &= \text{Sc} \left\{ \left[\underline{D} \frac{H[f]}{|H[f]|} \right] \sin \theta(\underline{x}) \cos \theta(\underline{x}) \right\} + \text{Sc} \left\{ [\underline{D}\theta(\underline{x})] \frac{H[f]}{|H[f]|} \right\}. \end{aligned}$$

In fact,

$$\begin{aligned} &\underline{D}f^+(\underline{x}) \\ &= \underline{D} \left\{ A(f) \left[\cos \theta(\underline{x}) + \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right] \right\} \\ &= \underline{D}A(f) \left[\cos \theta(\underline{x}) + \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right] + A(f) \underline{D} \left[\cos \theta(\underline{x}) + \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right]. \end{aligned}$$

$$\underline{D} \left[\cos \theta(\underline{x}) + \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right] = -\sin \theta \underline{D}\theta + \cos \theta \underline{D}\theta \frac{H[f]}{|H[f]|} + \underline{D} \left(\frac{H[f]}{|H[f]|} \right) \sin \theta.$$

Then

$$\begin{aligned} &\underline{D}f^+(\underline{x})[f^+(\underline{x})]^{-1} \\ &= \underline{D}f^+(\underline{x}) \left\{ \frac{1}{A(f)} \left[\cos \theta(\underline{x}) - \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right] \right\} \\ &= \frac{\underline{D}A(f)}{A(f)} + \left[\underline{D}e^{[\frac{H[f]}{|H[f]|} \theta(\underline{x})]} \right] \left[e^{[\frac{H[f]}{|H[f]|} \theta(\underline{x})]} \right]^{-1} \\ &= \frac{\underline{D}A(f)}{A(f)} + \underline{D} \left[\cos \theta(\underline{x}) + \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right] \left[\cos \theta(\underline{x}) - \frac{H[f]}{|H[f]|} \sin \theta(\underline{x}) \right]. \end{aligned}$$

Therefore,

$$\begin{aligned}
& \text{Sc}\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\} \\
= & \text{Sc}\left\{\left[\underline{D}e^{\frac{H[f]}{|H[f]|}\theta(\underline{x})}\right]\left[e^{\frac{H[f]}{|H[f]|}\theta(\underline{x})}\right]^{-1}\right\} \\
= & \text{Sc}\left\{\left[\underline{D}\frac{H[f]}{|H[f]|}\right]\sin\theta(\underline{x})\cos\theta(\underline{x})+\left[\underline{D}\theta(\underline{x})\right]\frac{H[f]}{|H[f]|}\right\} \\
= & \text{Sc}\left\{\left[\underline{D}\frac{H[f]}{|H[f]|}\right]\sin\theta(\underline{x})\cos\theta(\underline{x})\right\}+\text{Sc}\left\{\left[\underline{D}\theta(\underline{x})\right]\frac{H[f]}{|H[f]|}\right\}.
\end{aligned}$$

For $m = 1$ the first term of the last expression becomes zero and the second term reduces to the ordinary phase derivative. For a monogenic signal $f^+(\underline{x}) = f(\underline{x}) + H[f](\underline{x})$ we define the mean of the Fourier frequency to be

$$\langle \underline{t} \rangle = \int_{\mathbf{R}^m} |\underline{t}| |\widehat{f^+}(\underline{t})|^2 d\underline{t}.$$

Now we show, analogously with (4.4),

$$\langle \underline{t} \rangle = \int_{\mathbf{R}^m} \text{Sc}\left\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\right\} |f^+(\underline{x})|^2 d\underline{x}.$$

In fact,

$$\begin{aligned}
\langle \underline{t} \rangle &= \int_{\mathbf{R}^m} |\underline{t}| |\widehat{f^+}(\underline{t})|^2 d\underline{t} \\
&= \int_{\mathbf{R}^m} |\underline{t}| \widehat{f^+}(\underline{t}) \overline{\widehat{f^+}(\underline{t})} d\underline{t} \\
&= \int_{\mathbf{R}^m} |\underline{t}| \left[\frac{1}{2} \left(1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|} \right) \widehat{f}(\underline{t}) \right] \overline{\frac{1}{2} \left(1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|} \right) \widehat{f}(\underline{t})} d\underline{t} \\
&= \int_{\mathbf{R}^m} \mathbf{i} \underline{t} \left[\frac{1}{2} \left(1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|} \right) \widehat{f}(\underline{t}) \right] \overline{\frac{1}{2} \left(1 + \mathbf{i} \frac{\underline{t}}{|\underline{t}|} \right) \widehat{f}(\underline{t})} d\underline{t} \\
&= \int_{\mathbf{R}^m} \underline{D}\widehat{f^+}(\underline{t}) \overline{\widehat{f^+}(\underline{t})} d\underline{t} \\
&= \int_{\mathbf{R}^m} \underline{D}f^+(\underline{x}) \overline{f^+(\underline{x})} d\underline{x} \\
&= \int_{\mathbf{R}^m} [\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1} f^+(\underline{x}) \overline{f^+(\underline{x})} d\underline{x} \\
&= \int_{\mathbf{R}^m} [\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1} |f^+(\underline{x})|^2 d\underline{x} \\
&= \int_{\mathbf{R}^m} \text{Sc}\left\{[\underline{D}f^+(\underline{x})][f^+(\underline{x})]^{-1}\right\} |f^+(\underline{x})|^2 d\underline{x}.
\end{aligned}$$

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