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Pointwise Estimates for a Class of Singular Integrals and Higher Commutators

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§I. Pointwise and Weak-type Estimates

Denote $M(R^k)$, $M(R^k \times R^k)$ as the space of Lebesque measurable functions defined on R^k, $R^k \times R^k$, respectively. Let $L:H \to M(R^k \times R^k)$ be a linear operator defined on H, which is a linear subspace of $M(R^k)$. There exist the follwing conditions on L and H:

i) If U is a convex open set in R^k, x, $y \in U$ and $a \in H$, then $\chi_U \cdot a \in H$, and

$$L(a)(x,y) = L(\chi_U \cdot a)(x,y),$$

where χ_U denotes the characteristic function of set U.

ii) There exists an operator G: $H \to M(R^k)$, such that for every open set $V \subset R^k$,

$$G(\chi_V a) = \chi_V G(a).$$

iii) Denote Q as a cube in R^k, its sides are parallel to the axes, Λ_r^r as the Hardy-Littlewood maximal function of $|f|^r$, $r \in [1, \infty)$, and $\Lambda_\infty(f) = \|f\|_\infty$. Let $2Q$ be the double of Q, and

$$A(a,x,y) = \sup_{\substack{Q \ni x \\ y \bar{\in} 2Q}} \frac{1}{|Q|} \int_Q \frac{|x-y|}{|x-t|} |L(a)(x,y) - L(a)(t,y)| dt.$$

$$B(a,x,y) = \sup_{\substack{Q \ni x \\ y \bar{\in} 2Q}} \frac{1}{|Q|} \int_Q \frac{|x-y|}{|x-t|} |L(a)(x,y) - L(a)(y,t)| dt.$$

Then for a certain $r \in (1, \infty]$, and every $a \in H$, $b = G(a)$,

$$\Lambda_r(L(a)(x,\cdot))(x) \leqslant C\Lambda_r(\Lambda_1(b))(x) \qquad \text{a.e.} \tag{1.1}$$

$$\Lambda_r(L(a)(\cdot,y))(x) \leqslant C\Lambda_r(\Lambda_1(b))(x) \qquad \text{a.e.} \tag{1.2}$$

$$\Lambda_r(A(a,x,))(x) \leqslant C\Lambda_r(\Lambda_1(b))(x) \qquad \text{a.e.} \tag{1.3}$$

$$\Lambda_r(B(a,x,\cdot))(x) \leqslant C\Lambda_r(\Lambda_1(b))(x) \qquad \text{a.e.} \tag{1.4}$$

in which the constants C are independent of a.

Let function $K \in C^\infty(R^k \setminus \{0\})$, we shall refer to the following inequalities as the standard estimates (on the kernel K):

iv) For every $x \neq 0$,

$$|K(x)| \leqslant \frac{C}{|x|^k}, \qquad |\nabla K(x)| \leqslant \frac{C}{|x|^{k+1}}, \tag{1.5}$$

where C are constans.

Denote

$$T_\varepsilon(a,f)(x) = \int_{|x-y|>\varepsilon} L(a)(x,y)K(x-y)f(y)dy,$$

there exist the following conditions:

v) For a certain pair of $p_1 \in (1,\infty)$, $r_1 \in (1,\infty]$ such that $q_1^{-1} = p_1^{-1} + r_1^{-1} \in (1,\infty)$, and for every $f \in L^{p_1}(R^k)$, every $a \in H$, $b = G(a)$, $\varepsilon \in (0,\infty)$,

$$\| T_\varepsilon(a,f)\|_{q_1} \leqslant C\|b\|_{r_1} \cdot \|f\|_{p_1}, \tag{1.6}$$

where the constant C is independent of ε.

vi) With the notation as in v), the limite

$$T(a,f)(x) = \lim_{\varepsilon \to 0} T_\varepsilon(a,f)(x)$$

exists a.e., and

$$\| T(a,f)\|_{q_1} \leqslant C\|b\|_{r_1} \|f\|_{p_1}, \tag{1.7}$$

C is a constant.

Our main theorem is as follows.

Theorem 1. *With the notation as above, there follow*

1°. *With the conditions* i), ii), (1.1), (1.3), iv) *and one of two conditions* v) *and* vi) *we have*

$$M(a,f)(x) = \sup_{\varepsilon>0} |T_\varepsilon(a,f)(x)| \leqslant C(\Lambda_1(T(a,f))(x) + \Lambda_{r_1}(\Lambda_1(b))(x)\Lambda_{p_1}(f)(x), \qquad \text{a.e.,}$$

Where p_1, q_1, r_1 *are as in* v) $T(a,f)$ *is as in* vi) *or is a weak-star accomulation point of the bounded family of continuous linear functionals* $\{T_\varepsilon(a,f)\}_{\varepsilon>0} \subset (L^{q_1'})^*$.

2°. *Suppose the extra conditions* (1.2), (1.4) *are satisfied besides all the conditions in* 1°, *then for* p_0: $1 = p_0^{-1} + r_1^{-1}$, r_1 *is as in* 1°, $f \in L^{p_0}(R^k)$, *we nave*

$$|\{x \in R^k: M(a,f)(x) > \lambda\}| \leqslant C\frac{\|b\|_{r_1}\|f\|_{p_0}}{\lambda}.$$

3°. *With* $r_1 = \infty$ *in* 1°, *then for* $q \in (1, q_1)$, $f \in L^q(R^k)$, *there exists* $\| M(a,f)\|_q \leqslant C\|b\|_{r_1} \cdot \|f\|_q$.

the constants C in 1°, 2°, 3° *depend only on the dimension k and the constants appearing in* iii)—vi).

Proof. Since 1° implies the weak-type (p_1, q_1) of $M(a,.)$ see [2], remark 1), then 3° follows from 1°, 2° and Marcinkiewicz interpolation theorem. So we only need to prove 1° and 2°.

Proof of 1°. Suppose that the condition v) is satisfied. If otherwise the condition vi) is satisfied, the proof is even simpler. Fix $x \in R^k$, $\delta \in (0, \infty)$, denote $\chi_\delta = \chi_{S(x,\delta)}$, where $S(x,\delta)$ denotes the ball of center x and radius δ, then for $\varepsilon \in (0, \delta)$, we have

$$T_\varepsilon(a, f - \chi_\delta f) = T_\delta(a, f - \chi_\delta f). \tag{1.8}$$

By Banach - Alaoglu theorem, there is a sequence ε_n such that $\lim \varepsilon_n = 0$, and for all $f \in L^{p_1}(R^k)$, $T_{\varepsilon_n}(a,f)$ converges weak-star to a $T(a,f) \in L^{q_1}(R^k)$, and

$$\| T(a,f) \|_{q_1} \leqslant C \| b \|_{r_1} \| f \|_{p_1}.$$

Passing to the limit $\varepsilon = \varepsilon_n \to 0$ in (1.8), we have

$$T(a, f - \chi_\delta f) = T_\delta(a, f - \chi_\delta f),$$

and therefore, for $t \in S(x, \delta/2)$, a.e. x,

$$\begin{aligned} & T_\delta(a,f)(x) - T(a,f)(t) + T(a, \chi_\delta f)(t) \\ = & \int_{|x-y|>\delta} f(y)(L(a)(x,y)K(x-y) - L(a)(t,y)K(t-y))dy \\ = & \int_{|x-y|>\delta} f(y) \sum_{i=1}^{2} \Delta_i(a,x,y,t)dy \end{aligned}$$

where

$$\begin{aligned} \Delta_1 &= L(a)(x,y)(K(x-y) - K(t-y)), \\ \Delta_2 &= (L(a)(x,y) - L(a)(t,y))K(t-y). \end{aligned}$$

Because for $|x-y| > \delta$, $|t-x| < \delta/2$ the condition iv) gives that

$$|K(x-y) - K(t-y)| \leqslant C\delta |x-y|^{-k-1}$$

we have

$$|\Delta_1| \leqslant C\delta |L(a)(x,y)| |x-y|^{-k-1}.$$

Together with

$$\Delta_2 \leqslant C\delta \frac{|x-y|}{|x-t|} |L(a)(x-y) - L(a)(t,y)| |x-y|^{-k-1}$$

there follows

$$\begin{aligned} & |T_\delta(a,f)(x)| \leqslant |T(a,f)(t)| + |T(a,\chi_\delta f)(t)| \\ & + C\left(\int_{|x-y|>\delta} \frac{\delta |f(y) L(a)(x,y)|}{|x-y|^{k+1}} dy + \int_{|x-y|>\delta} \frac{\delta |f(y)| |x-y| |L(a)(x,y) - L(a)(t,y)|}{|x-y|^{k+1} \cdot |x-t|} dy \right). \end{aligned}$$

Integrating both sides of the last inequality in t over $S(x,\delta/2)$ and dividing by $|S(x,\delta/2)|$, we get

$$|T_\delta(a,f)(x)| \leqslant \Lambda_1(T(a,f))(x) + C\Big(\frac{1}{|S(x,\delta/2)|}\int_{S(x,\delta/2)} |T(a,\chi_\delta f)(t)|dt$$

$$+\int_{|x-y|>\delta} \frac{\delta|f(y)L(a)(x,y)|}{|x-y|^{k+1}}dy + \int_{|x-y|>\delta}\frac{\delta|f(y)|A(a,x,y)}{|x-y|^{k+1}}dy\Big)$$

$$= \Lambda_1(T(a,f))(x) + C\sum_{i=1}^{3} I_i.$$

For I_2 we have, by using (1.1)

$$I_2 = \sum_{i=0}^{\infty} \int_{2^i\delta<|x-y|\leqslant 2^{i+1}\delta} \frac{\delta|f(y)L(a)(x,y)|}{|x-y|^{k+1}}dy$$

$$\leqslant C\sum_{i=1}^{\infty}\frac{(2^{i+1}\delta)^k}{(2^i\delta)^{k+1}}\frac{1}{(2^{i+1}\delta)^k}\int_{|x-y|\leqslant 2^{i+1}\delta}|f(y)L(a)(x,y)|dy$$

$$\leqslant C\sum_{i=1}^{\infty}\frac{1}{2^i}\Big(\frac{1}{(2^{i+1}\delta)^k}\int_{|x-y|\leqslant 2^{i+1}\delta}|f(y)|^{r_1'}dy\Big)^{\frac{1}{r_1'}}\Big(\frac{1}{(2^{i+1}\delta)^k}\int_{|x-y|\leqslant 2^{i+1}\delta}|L(a)(x,y)|^{r_1}dy\Big)^{\frac{1}{r_1}}$$

$$\leqslant C\Lambda_{r_1}(\Lambda_1(b))(x)\Lambda_{r_1'}(f)(x) \leqslant C\Lambda_{r_1}(\Lambda_1(b))(x)\Lambda_{p_1}(f)(x) \quad \text{a.e.} \tag{1.9}$$

By the same method we can obtain the same estimate for I_3. To see I_1, from the condition i), $\forall\ \varepsilon\in(0,\delta)$, we have

$$T_\varepsilon(a,\chi_\delta f)(t) = \int_{\varepsilon<|t-y|<\delta} L(a)(t,y)K(t-y)f(y)dy$$

$$= \int_{\varepsilon<|t-y|<\delta} L(\chi_{2\delta}a)(t,y)K(t-y)f(y)dy$$

$$\doteq T_\varepsilon(\chi_{2\delta}a,\chi_\delta f)(t).$$

Passing to the limite, there follows

$$T(a,\chi_\delta f)(t) = T(\chi_{2\delta}a,\chi_\delta f)(t) \qquad \text{for} \quad t\in S(x,\delta/2).$$

By using Hölder inequality, we have

$$I_1 \leqslant \frac{C}{\delta^k}\int_{|t-x|<\delta/2}|T(a,\chi_\delta f)(t)|dt = \frac{1}{\delta^k}\int_{|t-x|<\delta/2}|T(\chi_{2\delta}a,\chi_\delta f)(t)|dt$$

$$\leqslant C\frac{\delta^{k/q_1'}}{\delta^k}\|T(\chi_{2\delta}a,\chi_\delta f)\|_{q_1}$$

$$\leqslant C\delta^{-k/q_1}\|\chi_{2\delta}b\|_{r_1}\|\chi_\delta f\|_{q_1} \leqslant C\Lambda_{r_1}(b)(x)\Lambda_{p_1}(f)(x) \qquad \text{a.e.}$$

Thus the proof of 1° is concluded.

Proof of 2°. We need the following lemma (for the proof see [2]).

Lemma 1. *If S is a sublinear operator of weak-type* (p_0, q_0), *a sufficient condition that S is also of weak-type* (p, q), *where* $p^{-1} - q^{-1} = p_0^{-1} - q_0^{-1}, p_0 > p \geqslant 1$, *is that for every sequece of pairwise disjoint cubes* Q_i, *which satisfies the Whithey decomposition condition:*

$$d(Q_i) \leqslant \mathrm{dist}(Q_i, (\bigcup Q_i)^c) \leqslant 4d(Q_i) \quad \text{for every } i,$$

and for every function h in $L^p(R^k)$ *having support in* $\bigcup_{Q_i}$ *such that*

$$\int_{Q_i} h(x)dx = 0 \qquad \textit{for every } i,$$

the following estimate holds

$$|\{x \in R^k:\ Q_i:\ S(h)(x) > \lambda\}| \leqslant C(\|h\|_p/\lambda)^q, \tag{1.10}$$

where $Q_i^* = 2Q_i$.

By applying Lemma 1 to the sublinear operator $M(a,f)$, which is known to be of weak-type (p_1, q_1), $q_1 > 1$, $q_1^{-1} - p_1^{-1} = r_1^{-1}$, we need to show that the condition (1.10) is satisfied.

Let $\{Q_i\}$, $h \in L^{p_0}(R^k)$ be as in Lemma 1, fix $x \in R^k \backslash \{0\}$ and $\varepsilon > 0$, denote

$$I(x, \varepsilon) = \{i: Q_i \bigcap S(x, \varepsilon) = \emptyset\},$$

$$J(x, \varepsilon) = \{i: Q_i \bigcap S(x, \varepsilon) \neq \emptyset, \quad Q_i \backslash S(x, \varepsilon) \neq \emptyset\},$$

then

$$T_\varepsilon(a, h)(x) = \sum_{i=1}^{\infty} \int_{Q_i \backslash S(x, \varepsilon)} \cdots dy = \sum_{i \in I(x,\varepsilon)} \int_{Q_i} \cdots dy + \sum_{i \in J(x,\varepsilon)} \int_{Q_i \backslash S(x,\varepsilon)} \cdots dy,$$

where each of the integrands is $L(a)(x, y)K(x - y)h(y)$. By the property of Whitney decomposition there are constants α, $\beta > 0$ such that for every $i \in J(x, \varepsilon)$, we have

$$Q_i \subset \{y: \alpha\varepsilon < |y - x| < \beta\varepsilon\},$$

so that

$$\begin{aligned}
&\sum_{i \in J(x,\varepsilon)} \int_{Q_i \backslash S(x,\varepsilon)} |L(a)(x,y)K(x, y)h(y)| dy \\
&\leqslant \int_{\alpha\varepsilon < |x - y| < \beta\varepsilon} \frac{|L(a)(x, y)|}{|x - y|^k} |h(y)| dy \\
&\leqslant C\left(\frac{1}{\varepsilon^k} \int_{|x-y| < \beta\varepsilon} |L(a)(x, y)|^{p_0'} dy\right)^{\frac{1}{p_0'}} \left(\frac{1}{\varepsilon^k} \int_{|x-y| < \beta\varepsilon} |h(y)|^{p_0} dy\right)^{\frac{1}{p_0}} \\
&\leqslant C\Lambda_{r_1}(\Lambda_1(b))(x)\Lambda_{p_0}(h)(x),
\end{aligned} \tag{1.11}$$

where we have used $r_1 = p_0'$.

For $i \in I(x, \varepsilon)$ we will show that

$$\left|\int_{Q_i} \cdots dy\right| = A_i(x) \leqslant C\delta_i \int_{Q_i} \frac{|L(a)(x,y)|}{|x-y|^{k+1}} |h(y)| dy + C\delta_i \int_{Q_i} \frac{B(a,x,y)}{|x-y|^{k+1}} |h(y)| dy \tag{1.12}$$

where $\delta_i = d(Q_i)$. Let $t \in Q_i$, since $\int_{Q_i} h(y)\ dy = 0$, we have

$$\int_{Q_i} L(a)(x,y)K(x-y)h(y)dy = \int_{Q_i} (L(a)(x,y)K(x-y) - L(a)(x,t)K(x-t))h(y)dy$$

$$= \int_{Q_i} h(y) \sum_{i=1}^{2} \Delta_i(a,x,y,t)dy \tag{1.13}$$

where

$$\Delta_1 = L(a)(x,y)(K(x-y) - K(x-t)),$$

$$\Delta_2 = (L(a)(x,y) - L(a)(x,t))K(x-t).$$

Integrating in t over Q_i both sides of (1.13), dividing by $|Q_i|$, as we did in the proof of 1° we get (1.12).

From (1.11), (1.12), there follows

$$M(a,h)(x) \leqslant C\Lambda_{r_1}(\Lambda_1(b))(x)\Lambda_{p_0}(h)(x) + \sum_{i=1}^{\infty} A_i(x).$$

The condition (1.10) will be satisfied if we show that

$$|\{x \in R^k : \Lambda_{r_1}(\Lambda_1(b))(x)\Lambda_{p_0}(h)(x) > \lambda\}| \leqslant C\frac{\|b\|_{r_1}\|h\|_{p_0}}{\lambda} \tag{1.14}$$

and

$$|\{x \in R^k \setminus \bigcup Q_i^* : \sum_{i=1}^{\infty} A_i(x) > \lambda\}| \leqslant C\frac{\|b\|_{r_1}\|h\|_{p_0}}{\lambda}. \tag{1.15}$$

For (1.14) see [2] Remark 1, it remains to show (1.15) only. In fact we have

$$\sum_{i=1}^{\infty} \int_{R^k \setminus \bigcup Q_i^*} A_i(x)\,dx \leqslant \sum_{i=1}^{\infty} \int_{R^k \setminus Q_i^*} A_i(x)dx.$$

There exists a constant γ, which depends only on the dimension k, such that if $x \bar{\in} Q_i^*$, $y \in Q_i$, then $|x-y| > \gamma\delta_i$. Thus, according to (1.2), (1.4), usin the same method in proving (1.9) we have

$$\int_{R^k\setminus Q_i^*} A_i(x)dx \leqslant C\int_{Q_i}\Bigg(\int_{|x-y|>\gamma\delta_i} \frac{\delta_i|L(a)(x,y)|}{|x-y|^{k+1}}dx\Bigg)|h(y)|dy$$

$$+ C\int_{Q_i}\Bigg(\int_{|x-y|>\gamma\delta_i} \frac{\delta_i B(a,x,y)}{|x-y|^{k+1}}dx\Bigg)|h(y)|dy$$

$$\leqslant C\int_{Q_i} \Lambda_1(\Lambda_1(b))(y)|h(y)|dy.$$

Therefore,

$$\sum_{i=1}^{\infty}\int_{R^k\setminus Q_i^*} A_i(x)dx \leqslant C\int_{R^k}\Lambda_1(\Lambda_1(b))(y)|h(y)|dy \leqslant C\|b\|_{r_1}\|h\|_{p_0}.$$

The proof is thus finished.

Theorem 1 has the following extension:

Theorem 2. *Suppose H_i, L_i, G_i, and K are as in Th. 1, where $i = 1,\cdots,n$. Denoting $a = (a_1,\cdots,a_n)$, $b = (G_1(a_1),\cdots,G_n(a_n))$ and*

$$L(a)(x,y) = \prod_{i=1}^{n} L_i(a_i)(x,y),$$

$$\|b\|_r = \prod_{i=1}^{n}\|b_i\|_{r_i},$$

$$\Lambda_r(\Lambda_1(b))(x) = \prod_{i=1}^{n}\Lambda_{r_i}(\Lambda_1(b_i))(x),$$

$$r = (r_1,\cdots,r_n),$$

where r_i' s satisfy one of the following conditions:

1°. $\forall\ i,\ r_i\in(1,\infty)$,

2°. $\forall\ i,\ r_i = \infty$.

If for q: $q^{-1} = p^{-1} + \sum_{i=1}^{n} r_i^{-1}$, for every i the conditions i) - iv) *and one of* v) *and* vi), *which is with respect to $p_1\in(1,\infty)$, are satisfied, then the conclusions of Th. 1 hold in the case of $p = p_1$, $q\in(1,\infty)$ for the conclusion 1°, $q = 1$ for the conclusion 2° and $q\in(1,q_1)$, $r_i = \infty$ for the conclusion 3°, respectively.*

The proof of Th. 2 is similar to the proof of Th. 1. We only point out following modification.

1°. To deal with the difference

$$L(a)(x,y) - L(a)(t,y)$$

we use the following formular:

$$\prod_{i=1}^{n} b_i - \prod_{i=1}^{n} a_i = \sum_{j=1}^{n} \left(\prod_{i=1}^{j-1} a_i\right)(b_j - a_j)\left(\prod_{k=j+1}^{n} b_k\right) \text{ with } \prod_{i=1}^{0} a_i = \prod_{k=n+1}^{n} b_k = 1.$$

2°. Instead of using Hölder inequality to two factors we use Hölder inquality to $n+1$ factors each time.

Remark 1. The condition iv) can be substituted by the following condition: $K(x) = \frac{\Omega(x)}{|x|^k}$, where Ω: $R^k\{0\} \to \mathbb{C}$ satisfies the conditions: Ω is homogeneous of degree 0, bounded, and

$$\frac{1}{|S(x,\delta)|} \cdot \int_{S(x,\delta)} |\Omega(x-y) - \Omega(t-y)| dt \leqslant C \frac{\delta}{|x-y|}, \quad \text{for } |x-y| > 2\delta.$$

§II. Application, Higher Commutators

Theorem 3. *with the notation as in Th. 2, let H_i be the Space of the functions whose all derivatives of order m_i belong to $L^{r_i}(R^k)$, $G_i(a_i) = \sum_{|\beta| = m_i} |\partial^\beta a_i|$, $L_i(a_i)(x,y) = \frac{P_{m_i}(a_i,x,y)}{|x-y|^{m_i}}$, where $P_{m_i}(a_i,x,y) = a_i(x) - \sum_{|\beta| < m_i} \frac{(\partial^\beta a_i)(y)}{\beta!}(x-y)^\beta$ for $m_i \in Z$, which is the set of positive integers, and $P_0(a_i,x,y) = a_i(x)$. Here $K(x) = \frac{\Omega(x)}{|x|^k}$, $\Omega(x)$ satisfies the conditions mentioned in Remark 1, and satisfies 1°. $\Omega(-x) = (-1)^{|m|+1}\Omega(x)$, $|m| = \sum_{i=1}^{n} m_i$, or 2°. $\int_{S^{k-1}} \Omega(x) x^\alpha d\sigma(x) =$ for $\forall$ α such that $|\alpha| \leqslant |m|$. Then the conclusions of Th. 2 hold.*

Proof of Theorem 3. It is easy to see that for all i, conclusions i), ii) are satisfied. vi) follows from the main results of [4]. In order to use Th. 2 (exactly, Remark 1), we only need to examine iii). The following lemma is needed.

Lemma 2. $\frac{|P_m(a,x,y)|}{|x-y|^m} \leqslant C(\Lambda_1(|\nabla^m a|)(x) + \Lambda_1(|\nabla^m a|)(y))$, *where $m \in Z$, and all the partial derivatives of order m of $a \in M(R^k)$ are locally integrable.*

Proof. The argument is similar to [5], Lemma 5. In fact, there we obtain that

$$\frac{|P_m(a,x,y)|}{|x-y|^m} \leqslant I_1 + I_2,$$

where

$$I_1 \leqslant C\frac{1}{\varepsilon} \int_{|\xi| \leqslant 3\varepsilon} \frac{|\nabla^m a(y-\xi)|}{|\xi|^{k-1}} d\xi,$$

$$I_2 \leqslant C\frac{1}{\varepsilon^m} \int_{|\xi| \leqslant 2\varepsilon} |u|^{m-k} |\nabla^m a(x-u)| du.$$

Using the method in proving (1.9), we obtain that

$$I_1 \leqslant C\Lambda_1(|\nabla^m a|)(y), \qquad I_2 \leqslant C\Lambda_1(|\nabla^m a|)(x).$$

By using the lemma, it is easy to prove (1.1), (1.2), so we only need to prove (1.3) and (1.4).

Proof of (1.3). For $x, t \in Q$, $y \bar{\in} 2Q$, we have

$$\frac{1}{|Q|}\int_Q \frac{|x-y|}{|x-t|}\left|\frac{P_m(a,x,y)}{|x-y|^m} - \frac{P_m(a,t,y)}{|t-y|^m}\right| dt$$

$$\leqslant \frac{1}{|Q|}\int_Q \frac{|x-y|}{|x-t|}\left|\frac{1}{|x-y|^m} - \frac{1}{|t-y|^m}\right| |P_m(a,x,y)|\, dt$$

$$+ \frac{1}{|Q|}\int_Q \frac{|x-y|}{|x-t|}\frac{1}{|t-y|^m}|P_m(a,x,y) - P_m(a,t,y)|\, dt$$

$$= I_1 + I_2,$$

where

$$I_1 \leqslant C(\Lambda_1(|\nabla^m a|)(x) + \Lambda_1(|\nabla^m a|)(y)).$$

To see I_2, using the formular

$$P_m(a,x,y) - P_m(a,t,y) = \int_0^1 \nabla_x P_m(a, x - s(x-t), y)\cdot(x-t)\, ds$$

and

$$\nabla_x P_m(a,x,y) = P_{m-1}(\nabla a, x, y),$$

which is a vector valued equality, we have

$$I_2 \leqslant \int_0^1 ds \frac{1}{|Q_{x,s}|}\int_{Q_{x,s}} \frac{|P_{m-1}(\nabla a, z, y)|}{|z-y|^{m-1}}\, dz$$

where $Q_{x,s} = x - s(x-Q)$, $x \in Q_{x,s}$.

Therefore

$$I_2 \leqslant C\int_0^1 ds \frac{1}{|Q_{x,s}|}\int_{Q_{x,s}} (\Lambda_1(|\nabla^m a|)(z) + \Lambda_1(|\nabla^m a|)(y))\, dz$$

$$\leqslant C(\Lambda_1(\Lambda_1(|\nabla^m a|))(x) + \Lambda_1(|\nabla^m a|)(y)),$$

and so $A(a,x,y)$ have the same estimate. Hence for all Q_1 and each $x \in Q_1$, we conclude

$$\left(\frac{1}{|Q_1|}\int_{Q_1} A(a,x,y)^p dy\right)^{\frac{1}{p}} \leqslant C\Lambda_p(\Lambda_1(|\nabla^m a|))(x).$$

Proof of (1.4). Now we use the vector valued inequality

$$\nabla_\xi P_m(a,x) = \frac{-1}{(m-1)!}\left(\sum_{j=1}^{k}(x_j-\xi_j)\frac{\partial}{\partial \xi_j}\right)^{m-1}\nabla a(\xi),$$

there follows

$$P_m(a,y,x)-P_m(a,y,t)=\int_0^1 \frac{-1}{(m-1)!}\left(\sum_{j=1}^{k}(x_j-\xi_j)\frac{\partial}{\partial \xi_j}\right)^{m-1}\nabla a(\xi)\bigg|_{\xi=x-s(x-t)}\cdot(x-t)ds$$

Therefore

$$\begin{aligned} I_2 &\leqslant C\int_0^1 ds\frac{1}{|Q|}\int_Q |(\nabla^m a)(x-s(x-t))|dt \\ &\leqslant C\int_0^1 ds\frac{1}{|Q_{x,s}|}\int_{Q_{x,s}} |(\nabla^m a)(z)|dz \\ &\leqslant C\Lambda_1(|\nabla^m a|)(x). \end{aligned}$$

The proof is thus finished.

Remark 2. In virtue of condition v), in Th. 3, the extra condition 1° or 2° upon $K(x)$ can be substituted by some weaker conditions. For example, when $n=1$, neither of the two conditions are necessary (see [3]). □

We turn to the higher commutators of multiplier operators.

Let $m=(m_1,\cdots,m_n)\in(Z\bigcup\{0\})^n$, $\alpha=(\alpha_1,\cdots,\alpha_n)\in(R^k)^n$,

$$R^{(m)}_{(-\alpha)} = R^{m_1}_{-\alpha_1}\cdots R^{m_n}_{-\alpha_n},$$

$$R^{m_i}_{-\alpha_i}g(\xi)=g(\xi-\alpha_i)-\sum_{|\beta|<m_i}\frac{\partial^\beta g(\xi)}{\beta!}(-\alpha_i)^\beta,$$

$$R^0_{-\alpha_i}g(\xi)=g(\xi-\alpha_i), \qquad \forall\ i.$$

Denote

$$M^l=\{\omega\in C^\infty(R^k\setminus\{0\}):\ \forall\ \ \beta,\exists\ C_\beta\ \text{ such that } |\partial^\beta\omega(\xi^\varepsilon)|\leqslant C_\beta|\xi|^{l-|\beta|}\},$$

and for $a=(a_1,\cdots,a_n)$, $a_i\in S(R^k)$, define

$$T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a,f)(x)=\int_{(R^k)^{n+1}} e^{ix\xi}R^{(m)}_{(-\alpha)}\omega(\xi)\hat{a}(\alpha)\hat{f}(\xi-[\alpha])d\alpha d\xi,$$

where $\hat{a}(\alpha)=\prod_{i=1}^{n}\hat{a}_i(\alpha_i)$, $[\alpha]=\sum_{i=1}^{n}\alpha_i$, $d\alpha=d\alpha_1\cdots d\alpha_n$, and denote $m+1=(m_1+1,\cdots,m_n+1)$, we have

Theorem 4. *If* $\omega\in M^l$, $l=|m|=\sum_{i=1}^{n}m_i$, *then*

1°. $\| T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a,f)\|_q \leqslant C \prod_{i=1}^{n} \|\nabla^{m_i} a_i\|_{r_i} \cdot \|f\|_p,$

where $q^{-1} = p^{-1} + \sum_{i=1}^{n} r_i^{-1}$, $p,q,r_i \in (1,\infty)$, $\forall\ i$.

2°. $|\{x: |T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a,f)(x)| > \lambda\}| \leqslant C\frac{1}{\lambda}\prod_{i=1}^{n} \|\nabla^{m_i} a_i\|_{r_i} \cdot \|f\|_p,$

where $1 = p^{-1} + \prod_{i=1}^{n} r_i^{-1}$, p, $r_i \in (1,\infty)$, $\forall\ i$.

3°. $\| T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a,f)\|_p \leqslant C \prod_{i=1}^{n} \|\nabla^{m_i} a_i\|_{\mathrm{BMO}} \|f\|_p,$

where $p \in (1,\infty)$, *and in every case* C *is a constant independent of* a, f.

Proof. 1° is a known result ([6], Th. 1). To prove 2° choose $\varphi \in C_0^\infty(R^k)$ such that supp $\varphi \subset \{1/2 \leqslant |\xi| \leqslant 2\}$, $\sum_{-\infty}^{\infty} \varphi(2^{-j}\xi) = 1$ for $\xi \neq 0$. Let $\omega_N(\xi) = \sum_{-N}^{N} \omega(\xi)\varphi(2^{-j}\xi)$, $K_N = (\omega_N)^\vee$, which denotes the inverse Fourier transformation of ω_N. By a standard argument we get

$$|K_N(x)| \leqslant \frac{C}{|x|^{k+l}}, \qquad |\nabla K_N(x)| \leqslant \frac{C}{|x|^{k+l+1}}, \tag{2.1}$$

where C are independent of N.

Denote

$$T_N^{(m)}(a,f) = T_{R^{(m)}_{(-\alpha)}\omega_N(\xi)}(a,f),$$

by a known result ([7]], Th. 1)

$$T_N^{(m)}(a,f)(x) = \int_{Rk} \prod_{i=1}^{n} \frac{P_{m_i}(a_i, x, y)}{|x-y|^{m_i}} K_N(x-y)|x-y|^l f(y)\ dy.$$

From conclusion 1° of the theorem (1.7) holds for $T_N^{(m)}$ and

$$T_N^{(m)}(a,f)(x) = \lim_{\varepsilon \to 0} (T_N^{(m)})_\varepsilon (a,f)(x), \quad x \in R^k, \tag{2.2}$$

so the condition vi) is satisfied. By using Th. 3 we obtain

$$|\{x: |T_N^{(m)}(a,f)(x)| > \lambda\}| \leqslant C\frac{1}{\lambda}\prod_{i=1}^{n} \|\nabla^{m_i} a_i\|_{r_i} \|f\|_p \tag{2.3}$$

where the constant C is independent of N.

From (2.2), we have

$$\{x: |T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a,f)(x)| > \lambda\} \subset \bigcup_{i=1}^{\infty} \bigcap_{N \geqslant i} \{x: |T_N^{(m)}(a,f)(x)| > \lambda\},$$

and thus the conclusion 2° holds.

To prove 3°, first, we have

$$R^{(m+1)}_{(-\alpha)}\omega(\xi) = \sum_{\substack{\bar{m}=(m_{i_1}+1,\cdots,m_{i_s}+1)\\ i_1<\cdots<i_s,\,0\leqslant s<n.\\ |[B]|=\sum_{i\neq i_j} m_i,\ (-\alpha')^B=\prod_{i\neq i_j}(-x_i)^{B_i}\\ B=(B_{j_1},\cdots,B_{j_l}),\ j_r\neq i_j.}} C_{\bar{m},B} R^{(\bar{m})}_{(-\alpha)}\partial^{[B]}\omega(\xi)(-\alpha')^B + R^{(m)}_{(-\alpha)}\omega(\xi), \tag{2.4}$$

and then by using the induction on n we conclude that (1.7) holds for $T_N^{(m+1)}$. So, from Th. 3 we get the weak-type estimate for the maximal operator of $T_N^{(m+1)}$, together with the property (2.1) of kernel $K(x)$. By using the same method as in [5], 3° holds for $T_N^{(m+1)}$ with a constant independent of N, then by Fatou' s lemma we conclude 3° for $T_{R^{(m+1)}_{(-\alpha)}}(a,f)$. □

A partial extension of Th. 4 is as follows.

Theorem 5. *For $\omega\in M^l$ and $\gamma_i\in(Z\bigcup\{0\})^k$, $i=1,2$, such that $l+|\gamma_1|+|\gamma_2|=|m|$, $|\gamma_1|\leqslant \min_{1\leqslant i\leqslant n}\{m_i\}$, then exist*

1°. $\|\partial^{\gamma_1}T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a,\partial^{\gamma}{}_2 f)\|_q\leqslant C\prod_{i=1}^{n}\|\nabla^{m_i}a_i\|_{r_i}\cdot\|f\|_p$, *where $p,q\in(1,\infty)$, $\forall\ i$, $r_i\in(1,\infty)$ or $\forall\ i$, $r_i=\infty$, $q^{-1}=p^{-1}+\sum_{i=1}^{n}r_i^{-1}$.*

2°. $|\{x: |\partial^{\gamma_1}T_{R^{(m)}_{(-\alpha)}\omega(\xi)}(a,\partial^{\gamma_2}f)(x)|>\lambda\}|\leqslant C\frac{1}{\lambda}\prod_{i=1}^{n}\|\nabla^{m_i}a_i\|_{r_i}\cdot\|f\|_p$, *where $p\in[1,\infty)$, $\forall\ i$, $r_i\in(1,\infty)$ or $\forall\ i$, $r_i=\infty$, $1=p^{-1}+\sum_{i=1}^{n}r_i^{-1}$. And in every case the constant C is independent of a, f.*

Proof. In the case of $r_i\in(1,\infty)$, $\forall\ i$, the inequality in 1° is a known result ([6], Th. 2). For the rest part of 1°, according to 3° of Th. 4 and equation (2.4), we have the inequality in $\gamma_1=\gamma_2=0$. By means of an induction on (γ_1,γ_2) (see [6], Th. 2) we obtain the inequality in general case.

To prove 2°, as before, we use the induction for the first case of r_i with the starting inequality in $\gamma_1=\gamma_2=0$, which comes from 2° of Th. 4. For the second case of r_i, when $\gamma_1=\gamma_2=0$, using 1° to $T_N^{(m)}(a,.)$, together with (2.1), we conclude that $T_N^{(m)}(a,.)$ is a Calderón-Zygmund operator, so the weak-type inequality holds with a constant C independent of N. Passing to the limite $N\to\infty$, we get the conclusion. For the general case we use the induction too.

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