Some Remarks on the Boundary Behaviors of the Hardy Spaces

Tao Qian and Jinxun Wang

In memory of Jaime Keller

Abstract. Some estimates and boundary properties for functions in the Hardy spaces are given.

Mathematics Subject Classification (2010). Primary 30G35, 31B05; Secondary 42B30, 31B25.

Keywords. monogenic, monogenic Hardy space, harmonic Hardy space, Cauchy's estimate.

1. Introduction

Let $\mathbb{D} = \{z = x + iy \in \mathbb{C} : |z| < 1\}$ be the open unit disc. The holomorphic Hardy space $\mathcal{H}^p(\mathbb{D})$ $(1 \le p < \infty)$ consists of all functions f that are holomorphic in $\mathbb{D}$ and satisfy

$$\|f\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta \right)^{1/p} < \infty.$$

Getting close to the boundary of $\mathbb{D}$, singularities may happen for functions in $\mathcal{H}^p(\mathbb{D})$, where we have the well known estimate (cf. [3])

$$(1 - |z|)^{1/p} |f(z)| \le C_p \|f\|_p \quad \text{for } 1 \le p < \infty.$$

By using the density of the holomorphic polynomials (cf. [9]), or that of the Poisson integrals (cf. [5]), one can prove that

$$\lim_{|z| \to 1^-} (1 - |z|)^{1/p} |f(z)| = 0 \quad \text{for } 1 \le p < \infty,$$

which is more precise than the previous inequality near the boundary.

This work was supported by Macao FDCT 056/2010/A3 and research grant of the University of Macau No. UL017/08-Y4/MAT/QT01/FST.

In the case $p = 2$, $\mathcal{H}^p(\mathbb{D})$ is of particular importance. It is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_0^{2\pi} f(e^{i\theta})\overline{g}(e^{i\theta})d\theta, \quad f, g \in \mathcal{H}^2(\mathbb{D}).$$

In a number of practical applications as the underlying space $\mathcal{H}^2(\mathbb{D})$ plays an important role (e.g., in signal processing, image processing and coding theory). Observing that for any function $f \in \mathcal{H}^2(\mathbb{D})$, we have

$$\langle f, \phi_a\rangle = \sqrt{1-|a|^2} f(a),$$

where $\phi_a(z) = \frac{\sqrt{1-|a|^2}}{1-\overline{a}z}$ is a unit vector of $\mathcal{H}^2(\mathbb{D})$ with the parameter $a \in \mathbb{D}$. By the aforementioned property, we get

$$\lim_{|a|\to 1^-} |\langle f, \phi_a\rangle| = 0,$$

which implies that there exists $a^\star \in \mathbb{D}$ such that $|\langle f, \phi_a^\star\rangle|$ attains the maximum value. This is crucial for the signal adaptive decomposition methods (as a variation and realization of greedy algorithm) introduced in [5, 8].

In this note, we give a generalization of the above result to higher dimensions, of which the special cases have been applied to the adaptive decomposition of functions of several variables ([6, 7]). Our method is a modification of the classic method (see [3, page 18]), which depends on some more delicate estimates. Before we state our main results, let us first have a quick review of some basic knowledge on Clifford algebra and Clifford analysis.

Let $e_1, \ldots, e_m$ be basic elements satisfying $e_ie_j + e_je_i = -2\delta_{ij}, i, j = 1, \ldots, n$, where δ_{ij} equals 1 if $i = j$ and 0 otherwise. Let $\mathbb{R}^{m+1} = \{x = x_0 + x_1e_1 + \cdots + x_me_m : x_i \in \mathbb{R}, 0 \le i \le m\}$ be identified with the usual $(m+1)$-dimensional Euclidean space. The real Clifford algebra generated by $e_1, \ldots, e_m$, denoted by $\mathscr{A}_m$, is an associative algebra in which each element is of the form $x = \sum_T x_Te_T$, where $x_T \in \mathbb{R}$, $e_T = e_{i_1}e_{i_2}\cdots e_{i_l}$ and $T = \{1 \le i_1 < i_2 < \cdots < i_l \le m\}$ runs over all ordered subsets of $\{1, \ldots, m\}$ and $x_\emptyset = x_0$, $e_\emptyset = e_0 = 1$. The norm and the conjugate of x are defined by $|x| = (\sum_T |x|_T^2)^{1/2}$ and $\overline{x} = \sum_T x_T\overline{e_T}$ respectively, where $\overline{e_T} = \overline{e_{i_l}}\cdots\overline{e_{i_2}}\,\overline{e_{i_1}}$ and $\overline{e_i} = -e_i$ for $i \ne 0$, $\overline{e_0} = e_0$. We have for any $x, y, z \in \mathscr{A}_m$, $\overline{xy} = \overline{y}\,\overline{x}$, $(xy)z = x(yz)$ and $|xy| \le 2^{m/2}|x||y|$.

A function $f(x) = \sum_T f_T(x)e_T \in C^1(\Omega, \mathscr{A}_m)$ is said to be *left monogenic* in the open set $\Omega \subset \mathbb{R}^{m+1}$ if and only if it satisfies the generalized Cauchy-Riemann equation

$$Df = \sum_{i=0}^m e_i\frac{\partial f}{\partial x_i} = 0,$$

where the Dirac operator D is defined by $D = \frac{\partial}{\partial x_0} + \nabla = \sum_0^m e_i\frac{\partial}{\partial x_i}$. If f is left monogenic, then each component of f is a real-valued harmonic function. For more information about the monogenic function theory, see [2].

Let $\mathbb{B}^m(x, \rho) = \{y \in \mathbb{R}^{m+1} : |y-x| < \rho\}$ be the open ball in $\mathbb{R}^{m+1}$, which is centered at x and of radius ρ. For simplicity, we denote $\mathbb{B}^m = \mathbb{B}^m(0, 1)$.

The monogenic Hardy space $\mathcal{H}^p(\mathbb{B}^m)$ $(1 \le p < \infty)$, consists of all functions f that are left monogenic in $\mathbb{B}^m$ and satisfy

$$\|f\|_p = \sup_{0<r<1} \left(\int_{|\eta|=1} |f(r\eta)|^p dS \right)^{1/p} < \infty, \tag{1.1}$$

where dS is the area element of $\partial\mathbb{B}^m$. We prove that

Theorem 1.1. *If $f \in \mathcal{H}^p(\mathbb{B}^m)$ $(1 \le p < \infty)$, then*

$$(1-|x|)^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| \le C_{m,p,|\alpha|} \|f\|_p, \tag{1.2}$$

where $\alpha = (l_0, l_1, \dots, l_m)$, $|\alpha| = \sum_{i=0}^m l_i$ and $\partial^\alpha = \partial_{x_0}^{l_0} \partial_{x_1}^{l_1} \cdots \partial_{x_m}^{l_m}$. Write $x = |x|\xi = r\xi$, then we have

$$\lim_{r \to 1^-} (1-|x|)^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| = 0 \tag{1.3}$$

uniformly in $|\xi| = 1$.

Corresponding to this, we also prove some propositions for the monogenic Hardy space $\mathcal{H}^p(\mathbb{R}_+^{m+1})$, which consists of all functions f that are left monogenic on the half space $\mathbb{R}_+^{m+1} = \{x = x_0 + \underline{x} \in \mathbb{R}^{m+1} : x_0 > 0, \underline{x} = x_1 e_1 + \cdots + x_m e_m \in \mathbb{R}^m\}$ and satisfy

$$\|f\|_p = \sup_{x_0>0} \left(\int_{\mathbb{R}^m} |f(x_0 + \underline{x})|^p d\underline{x} \right)^{1/p} < \infty, \tag{1.4}$$

where $d\underline{x} = dx_1 \cdots dx_m$. We note that for $f \in \mathcal{H}^p(\mathbb{R}_+^{m+1})$ $(1 \le p < \infty)$, the boundary values $f(\underline{x}) = \lim_{x_0 \to 0^+} f(x_0 + \underline{x})$ exist almost everywhere and comprise a function in $L^p(\mathbb{R}^m)$, of which the Poisson integral coincides with f ([4]).

Theorem 1.2. *Suppose $f \in \mathcal{H}^p(\mathbb{R}_+^{m+1})$ $(1 \le p < \infty)$, then*

$$x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| \le C_{m,p,|\alpha|} \|f\|_p; \tag{1.5}$$

moreover,

$$\lim_{x_0 \to 0^+} x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x_0 + \underline{x})| = \lim_{x_0 \to +\infty} x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x_0 + \underline{x})| = 0 \tag{1.6}$$

holds uniformly with respect to $\underline{x} \in \mathbb{R}^m$, and

$$\lim_{|\underline{x}| \to +\infty} x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x_0 + \underline{x})| = 0 \tag{1.7}$$

holds uniformly in $x_0 > 0$.

Remark 1.3. Similar discussions as in Section 2 will show that Theorem 1.1 (resp. Theorem 1.2) holds for the harmonic Hardy space $H^p(\mathbb{B}^m)$ (resp. $H^p(\mathbb{R}_+^{m+1})$) for $1 < p < \infty$, where by definition, a function f lies in $H^p(\mathbb{B}^m)$ (resp. $H^p(\mathbb{R}_+^{m+1})$) means that f is harmonic in $\mathbb{B}^m$ (resp. $\mathbb{R}_+^{m+1}$) and (1.1) (resp. (1.4)) holds. But for the case $p = 1$, (1.3) (resp. (1.6) and (1.7)) may not hold for $H^1(\mathbb{B}^m)$ (resp. $H^1(\mathbb{R}_+^{m+1})$). For example, $f(x_0, x_1) = \frac{x_0}{x_0^2 + x_1^2} \in H^1(\mathbb{R}_+^2)$, but $x_0 f(x_0, x_1)$ does not uniformly tend to zero as $x_0 \to 0^+$.

2. Proof of the Theorems

Proof of Theorem 1.1. From Cauchy's estimate (cf. [1]) we know that

$$|\partial^\alpha f(x)| \le C_{m,|\alpha|}(1-|x|)^{-|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,\frac{1-|x|}{2})} |f(y)|,$$

hence

$$(1-|x|)^{|\alpha|+\frac{m}{p}}|\partial^\alpha f(x)| \le C_{m,|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,\frac{1-|x|}{2})} (1-|y|)^{\frac{m}{p}}|f(y)|.$$

So, to prove (1.2) and (1.3), it is enough to show that

$$(1-|x|)^{\frac{m}{p}}|f(x)| \le C_{m,p}\|f\|_p \tag{2.1}$$

and

$$\lim_{r\to 1^-}(1-|x|)^{\frac{m}{p}}|f(x)| = 0 \tag{2.2}$$

for $1 \le p < \infty$.

Denote by $V_r = C_m(1-r)^{m+1}$ the volume of the ball $\mathbb{B}^m(x, 1-r)$, write $y = |y|\eta = \rho\eta$, note that

$$|y-x| \ge ||y|-|x|| = |\rho - r|,$$

and

$$\begin{aligned}|y-x| &= |\rho\eta - r\xi|\\ &= |r(\eta-\xi)-(r-\rho)\eta|\\ &\ge r|\eta-\xi| - |r-\rho|\\ &\ge r|\eta-\xi| - |y-x|,\end{aligned}$$

so $y \in \mathbb{B}^m(x, 1-r)$ implies

$$\begin{cases}2r-1 < \rho < 1,\\ |\eta-\xi| < 2(1-r)/r.\end{cases}$$

Hence, for $1 \le p < \infty$, we have

$$\begin{aligned}&|(1-r)^{m/p}f(x)|\\ &\quad= (1-r)^{m/p}\Big|V_r^{-1}\int_{\mathbb{B}^m(x,1-r)} f(y)dy\Big|\\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}\int_{\mathbb{B}^m(x,1-r)} |f(y)|^p dy\Big)^{1/p}\\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}\int_{2r-1}^1 \rho^m \int_{|\eta-\xi|<2(1-r)/r} |f(\rho\eta)|^p dS d\rho\Big)^{1/p}\\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}2(1-r)\sup_{0<\rho<1}\int_{|\eta-\xi|<\frac{2(1-r)}{r}} |f(\rho\eta)|^p dS\Big)^{1/p} \qquad (2.3)\\ &\quad\le (1-r)^{m/p}\Big(V_r^{-1}2(1-r)\sup_{0<\rho<1}\int_{|\eta|=1} |f(\rho\eta)|^p dS\Big)^{1/p}\\ &\quad= C_{m,p}\|f\|_p.\end{aligned}$$

(2.1) is now proved. On the other hand,

$$(2.3) \leq C_{m,p}\Big(\int_{|\eta-\xi|<\frac{2(1-r)}{r}} \sup_{0<\rho<1} |f(\rho\eta)|^p dS\Big)^{1/p}.$$

Note that as a function of η, $\sup_{0<\rho<1}|f(\rho\eta)| \in L^p(\partial\mathbb{B}^m)$, and the measure of the set $\{\eta : |\eta-\xi| < 2(1-r)/r\}$ tends to zero as $r \to 1^-$, (2.2) follows by the absolute continuity of the Lebesgue integral. □

Proof of Theorem 1.2. By Cauchy's estimate we have

$$|\partial^\alpha f(x)| \leq C_{m,|\alpha|} x_0^{-|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,x_0/2)} |f(y)|,$$

hence

$$x_0^{|\alpha|+\frac{m}{p}} |\partial^\alpha f(x)| \leq C_{m,|\alpha|} \max_{y\in\partial\mathbb{B}^m(x,x_0/2)} y_0^{m/p}|f(y)|.$$

So, the proof of (1.5) and (1.6) is now reduced to the proof of the following

$$x_0^{m/p}|f(x)| \leq C_{m,p}\|f\|_p \tag{2.4}$$

and

$$\lim_{x_0\to 0^+} x_0^{m/p}|f(x)| = \lim_{x_0\to+\infty} x_0^{m/p}|f(x)| = 0 \tag{2.5}$$

for $1 \leq p < \infty$. Once these have been proved, the proof of (1.7) will be reduced to the proof of

$$\lim_{|\underline{x}|\to+\infty} |f(x_0+\underline{x})| = 0 \tag{2.6}$$

uniformly with respect to $x_0 \in [a,b] \subset (0,+\infty)$.

Denote by $V_{x_0} = C_m x_0^{m+1}$ the volume of the ball $\mathbb{B}^m(x,\frac{x_0}{2})$, then for $1 \leq p < \infty$,

$$\begin{aligned} x_0^{m/p}|f(x)| &= x_0^{m/p}\Big|V_{x_0}^{-1}\int_{\mathbb{B}^m(x,\frac{x_0}{2})} f(y_0+\underline{y})dy\Big| \\ &\leq x_0^{m/p}\Big(V_{x_0}^{-1}\int_{\mathbb{B}^m(x,\frac{x_0}{2})} |f(y_0+\underline{y})|^p dy\Big)^{1/p} \qquad (2.7) \\ &\leq x_0^{m/p}\Big(V_{x_0}^{-1}\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}}\int_{\mathbb{R}^m} |f(y_0+\underline{y})|^p d\underline{y}dy_0\Big)^{1/p} \qquad (2.8) \\ &\leq x_0^{m/p}\Big(x_0 V_{x_0}^{-1}\sup_{y_0>0}\int_{\mathbb{R}^m} |f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p} \\ &= C_{m,p}\|f\|_p, \end{aligned}$$

so (2.4) is verified.

On the other hand, when x_0 is small,

$$
\begin{aligned}
(2.7) &\leq x_0^{m/p}\Big(V_{x_0}^{-1}\int_{\frac{x_0}{2}}^{\frac{3x_0}{2}}\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}|f(y_0+\underline{y})|^p d\underline{y}dy_0\Big)^{1/p}\\
&\leq x_0^{m/p}\Big(x_0V_{x_0}^{-1}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}\\
&\leq C_{m,p}\Big(\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}\\
&\leq C_{m,p}\Big(\int_{|\underline{y}-\underline{x}|\leq\frac{x_0}{2}}\sup_{y_0>0}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}.
\end{aligned}
$$

Note that as a function of $\underline{y}$, $\sup_{y_0>0}|f(y_0+\underline{y})|\in L^p(\mathbb{R}^m)$ and the measure of the set $\{\underline{y}:|\underline{y}-\underline{x}|\leq\frac{x_0}{2}\}$ tends to zero as $x_0\to0^+$, by the absolute continuity of the Lebesgue integral we have

$$\lim_{x_0\to0^+}x_0^{m/p}|f(x)|=0.$$

When x_0 is large,

$$
\begin{aligned}
(2.8) &\leq x_0^{m/p}\Big(x_0V_{x_0}^{-1}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}\int_{\mathbb{R}^m}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}\\
&\leq C_{m,p}\Big(\int_{\mathbb{R}^m}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|^p d\underline{y}\Big)^{1/p}
\end{aligned}
$$

holds uniformly with respect to $\underline{x}\in\mathbb{R}^m$, and

$$\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|\leq\sup_{y_0>0}|f(y_0+\underline{y})|\in L^p(\mathbb{R}^m)\quad\text{for } 1\leq p<\infty.$$

Also, from (2.4) we know that

$$\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|\leq x_0^{-m/p}C_{m,p}\|f\|_p,$$

which implies

$$\lim_{x_0\to+\infty}\sup_{y_0\in(\frac{x_0}{2},\frac{3x_0}{2})}|f(y_0+\underline{y})|=0$$

holds uniformly with respect to $\underline{y}\in\mathbb{R}^m$. By the Lebesgue's dominated convergence theorem we have

$$\lim_{x_0\to+\infty}x_0^{m/p}|f(x)|=0.$$

Now we proceed to prove (2.6). Since

$$\begin{aligned}
&|f(x_0+\underline{x})| \\
&\quad = \frac{\Gamma(\frac{m+1}{2})}{\pi^{\frac{m+1}{2}}}\Big| \int_{\mathbb{R}^m} \frac{x_0}{(|\underline{x}-\underline{y}|^2+x_0^2)^{\frac{m+1}{2}}} f(\underline{y})d\underline{y}\Big| \\
&\quad \le b\frac{\Gamma(\frac{m+1}{2})}{\pi^{\frac{m+1}{2}}} \int_{\mathbb{R}^m} \frac{|f(\underline{y})|d\underline{y}}{(|\underline{x}-\underline{y}|^2+a^2)^{\frac{m+1}{2}}} \\
&\quad = C_m\Big(\int_{|\underline{y}|>N} \frac{|f(\underline{y})|d\underline{y}}{(|\underline{x}-\underline{y}|^2+a^2)^{\frac{m+1}{2}}} + \int_{|\underline{y}|\le N} \frac{|f(\underline{y})|d\underline{y}}{(|\underline{x}-\underline{y}|^2+a^2)^{\frac{m+1}{2}}}\Big) \\
&\quad = C_m(I_1+I_2),
\end{aligned}$$

by Hölder's inequality,

$$\begin{aligned}
I_1 &\le \Big(\int_{|\underline{y}|>N} (|\underline{x}-\underline{y}|^2+a^2)^{\frac{-(m+1)p'}{2}} d\underline{y}\Big)^{1/p'} \Big(\int_{|\underline{y}|>N} |f(\underline{y})|^p d\underline{y}\Big)^{1/p} \\
&\le \Big(\int_{\mathbb{R}^m} (|\underline{y}|^2+a^2)^{\frac{-(m+1)p'}{2}} d\underline{y}\Big)^{1/p'} \Big(\int_{|\underline{y}|>N} |f(\underline{y})|^p d\underline{y}\Big)^{1/p} \\
&\le C_{m,p}\Big(\int_{|\underline{y}|>N} |f(\underline{y})|^p d\underline{y}\Big)^{1/p},
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$. Because $f(\underline{y}) \in L^p(\mathbb{R}^m)$, I_1 is small provided N is large enough. With N fixed,

$$I_2 \le \frac{C_m}{|\underline{x}|^{m+1}} \int_{|\underline{y}|\le N} |f(\underline{y})|d\underline{y} \to 0 \quad (|\underline{x}| \to +\infty),$$

due to $f(\underline{y})$ is integrable on $\{\underline{y} : |\underline{y}| \le N\}$, that proves (2.6).

The proof of Theorem 1.2 is now complete. □

References

[1] S. Axler, P. Bourdon, W. Ramey, *Harmonic Function Theory.* 2nd ed., GTM, Springer-Verlag, 2001.

[2] F. Brackx, R. Delanghe, F. Sommen, *Clifford Analysis.* London: Pitman Advanced Publishing Program, 1982.

[3] John B. Garnett, *Bounded Analytic Functions.* Revised 1st ed., GTM, Springer-Verlag, 2007.

[4] J. E. Gilbert, Margaret A. M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis.* Cambridge: Cambridge University Press, 1991.

[5] T. Qian, *Intrinsic mono-component decomposition of functions: An advance of Fourier theory.* Math. Meth. Appl. Sci., 33 (2010), 880–891.

[6] T. Qian, W. Sprößig, J. X. Wang, *Adaptive Fourier decomposition of functions in quaternionic Hardy spaces.* Math. Meth. Appl. Sci., to appear.

[7] T. Qian, J. X. Wang, Y. Yang, *Matching pursuits among shifted Cauchy kernels in higher-dimensional spaces.* Preprint.

[8] T. Qian, Y. B. Wang, *Adaptive Fourier series–a variation of greedy algorithm.* Advances in Computational Mathematics, 34 (2011), 279–293.

[9] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball.* GTM, Springer-Verlag, 2005.

Tao Qian
Department of Mathematics
Faculty of Science and Technology
University of Macau
Taipa, Macao, China
e-mail: fsttq@umac.mo

Jinxun Wang
Department of Mathematics
Faculty of Science and Technology
University of Macau
Taipa, Macao, China
e-mail: wjxpyh@gmail.com