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Higher Order Boundary Integral Formula and Integro-Differential Equation on Stein Mainfolds

Lüping Chen · Tongde Zhong · Tao Qian

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Abstract This paper deals with the boundary value properties and the higher order singular integro-differential equation. On Stein manifolds, the Hadamard principal value, the Plemelj formula and the composite formula for higher order Bochner–Martinelli type integral are given. As an application, the composite formula is used for discussing the solution of the higher order singular integro-differential equation.

Keywords Stein manifolds · Higher order singular integral · Bochner–Martinelli integral · Plemelj formula · Composite formula · Integro-differential equation

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1 Introduction

As it is know, the limit value formula plays an important role in the study of the singular boundary problem. The existence and multiplicity of the solutions for such

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L. Chen (✉)
School of Mathematical Sciences, Xiamen University, Xiamen 361005, Fujian, People's Republic of China
e-mail: lpchen@xmu.edu.cn

T. Zhong
e-mail: zhongtd@xmu.edu.cn

T. Qian
Faculty of Science and Technology, The University of Macau, Macau 3001, People's Republic of China
e-mail: fsttq@umac.mo

problems have received a great deal of attention. Early in 1957, Qikeng Lu and Tongde Zhong initiated the study of the boundary properties of the singular integral with Bochner–Martinelli kernel. They obtained the Plemelj formula of a singular integral on a bounded domain with $C^{(1)}$ smooth boundary in C^n [1,2]. Later, on a relatively compact domain with $C^{(1)}$ smooth boundary in Stein manifolds, Tongde Zhong obtained the Plemelj formula of the Bochner–Martinelli type singular integral [3]. In 2001, Tao Qian and Tongde Zhong gave a new Hadamard principal value definition for the higher order Bochner–Martinelli type singular integral on a closed smooth orientable manifold in C^n, they obtained the Plemelj formula and the composite formula [4] of the higher order singular integral. In 2008 and 2010, we have some works deal with the higher order singular integral and singular integral equation [5,6]. The aim of this article is to generalize the above results to the higher order singular integral on Stein manifolds. Since the structure of the manifold is complicated, we should use special techniques to solve an integro-differential equation on this area. Here, the key point of our process is to introduce the localization method and the biholomorphic mapping. Then, on Stein manifolds, the Plemelj formula and the composite formula for higher order Bochner–Martinelli type integral are given. As an application, the composite formula is used for discussing the solution of the higher order singular integro-differential equation.

This article is organized as follows: Sect. 2 is devoted to preliminary knowledge, notation and terminology. In Sect. 3 we devote ourselves to the Hadamard principal value, the Plemelj and the composite formula for higher order singular integral on Stein manifolds. In Sect. 4, we describe the regularization method, draw conclusions for higher order integro-differential equation on Stein manifolds.

2 Definition and Preliminary Knowledge

In this section, we introduce some necessary definitions for proving the main theorems of this paper in Sects. 3 and 4. It is known that Henkin and Leiterer [7] extended the famous Bochner–Martinelli formula from C^n to Stein manifolds. In this paper, what we shall be working with is a generic Stein manifold, denoted by M. Suppose that $T(M)$ is a complex tangent vector bundle and $T^*(M)$ is a complex cotangent vector bundle defined on M, $\tilde{T}(M \times M)$ and $\tilde{T}^*(M \times M)$ are pullbacks of $T(M)$ and $T^*(M)$. Under the pullbacks, $M \times M \to M$ and $(z, \xi) \to z$ respectively, holomorphic section $S(z, \xi) : M \times M \mapsto \tilde{T}(M \times M)$ generates the analytic sub-sheaf of sheaf ${}_{M\times M}\vartheta$, that is

(i) For every $z \in M$,

$$S(z, z) = 0.$$

The mapping

$$S(z, \cdot) : M \to T_z(M)$$

is biholomorphic in some of the neighborhoods of z.

(ii) For every $z \neq \xi$,

$$S(z,\xi) \neq 0.$$
$$\bar{S}(z,\xi) = \sigma S(z,\xi),$$

mapping

$$\sigma : \tilde{T}(M \times M) \to (\tilde{T})^*(M \times M)$$

is equivalent to the mapping in C^n.

Let D be a relatively compact domain, whose boundary is $C^{(1)}$ smooth or $C^{(1)}$ piecewise smooth. We adopt the following definition [7]:

Definition 2.1 [7] Suppose that $\nu \geq 2\kappa n$ and $(\bar{S}, \kappa)$ is a Leray section about (D, S, φ), the expression

$$\frac{(n-1)!}{(2\pi i)^n} \varphi^\nu(z,\xi) \frac{\omega'_\xi(\bar{S}(z,\xi) \wedge \omega_\xi(S(z,\xi))}{|S(z,\xi)|^{2n}_\sigma} \hat{=} \Omega(\varphi^\nu, \bar{S}, S) \tag{2.1}$$

is called a Bochner–Martinelli kernel on the Stein manifold,
where

$$\omega'_\xi(\bar{S}(z,\xi)) = \sum_{k=1}^{n} (-1)^{k-1} \bar{S}_k d\bar{S}_1 \wedge \cdots [d\bar{S}_k] \wedge \cdots d\bar{S}_n,$$
$$\omega_\xi(S(z,\xi)) = d_\xi S_1 \wedge \cdots \wedge d_\zeta S_n,$$

norm $|S(z,\xi)|_\sigma = \langle \bar{S}(z,\xi), S(z,\xi) \rangle^{\frac{1}{2}}$ is an Euclidean measure, $\varphi(z,\xi)$ is a holomorphic function on $M \times M$.

From the proof of the Theorem 4.12.1 in [7] and [8], obviously, we have the following results

(1) In $D \subset\subset M$, there are isolated singular points whose orders are $2n-1$.
(2) For every $z, \xi \in M$ and $\xi \neq z$, function $\varphi^k \parallel S \parallel_\sigma^{-2}$ is $C^{(2)}$.
(3) When integer $\nu \geq 2n\kappa$, for every fixed point $z \in M$ and $\xi \in M \backslash \{z\}$, formula (2.1) is $C^{(1)}$.
(4) For every $z \in M$, $\varphi(z,z) = 1$.

Definition 2.2 Suppose that D is a relative compact domain on Stein manifold M, $f(\xi)$ is a function defined on the boundary ∂D, then

$$F(\zeta) = \int_{\partial D_\xi} f(\xi) \Omega(\varphi^\nu, \bar{S}(\zeta,\xi), S(\zeta,\xi)), \quad \zeta \in \partial D \tag{2.2}$$

is called a singular integral. If $f(\xi)$ satisfies the Hölder continuity condition, the principal value of the integral exists. The Cauchy principal value can be defined as follows [3]

$$\text{V.P.}F(\zeta) = \lim_{\delta\to 0} \int\limits_{\sum_\delta(\zeta,\xi)} f(\xi)\Omega(\varphi^\nu, \bar{S}(\zeta,\xi), S(\zeta,\xi)), \quad \zeta \in \partial D, \tag{2.3}$$

where

$$\Sigma_\delta(\zeta,\xi) = \partial D_\xi - \sigma_\delta(\zeta,\xi),\ \sigma_\delta(\zeta,\xi) = \partial D_\xi \cap B_\delta(\zeta),\ B_\delta(\zeta) = \{z : |z-\zeta| < \delta\}.$$

3 The Hadamard Principal Value and the Composite Formula for Higher Order Singular Integrals on Stein Manifolds

In the following, we should first study the higher order Bochner–Martinelli type integral on Stein manifolds of the form

$$\int\limits_{\partial D} f(\xi)\frac{\bar{S}_k(z,\xi)}{|S(z,\xi)|^2}\Omega(\varphi^\nu(z,\xi)\bar{S}(z,\xi), S(z,\xi)), \tag{3.1}$$

where f is a differentiable function defined in a neighborhood of the boundary ∂D. On the boundary ∂D, the first-order derivatives of f satisfy the Hölder continuity condition, here the exponent $\alpha > 0$. For simplicity we denote this function class by $H_1(\alpha)$.

Lemma 3.1 *Suppose that $f \in H_1(\alpha)$, $B(\zeta,\delta)$ is a neighborhood of $\zeta \in \partial D$, for a sufficient small radius δ, we have*

$$\begin{aligned}
&\int\limits_{\partial D\setminus B(\zeta,\delta)} f(\xi)\frac{\bar{S}_k(\zeta,\xi)}{|S(\zeta,\xi)|^2}\Omega(\varphi^\nu(\zeta,\xi)\bar{S}(\zeta,\xi), S(\zeta,\xi))\\
&= -\frac{1}{n}C_n \int\limits_{\partial(\partial D\setminus B(\zeta,\delta))} [f(\xi)\varphi^\nu(\zeta,\xi)]\\
&\quad\times\frac{\sum_{j=1}^n (-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}\\
&\quad+\frac{1}{n}C_n \int\limits_{\partial D\setminus B(\zeta,\delta)} d[f(\xi)\varphi^\nu(\zeta,\xi)]\cdot\\
&\quad\times\frac{\sum_{j=1}^n (-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}},
\end{aligned} \tag{3.2}$$

where $C_n = \frac{(n-1)!}{(2\pi i)^n}$.

Proof To show the identity we first carry out some differential computation, as follows.

$$\begin{aligned}
&d\left\{\frac{\sum_{j=1}^n (-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}\right\}\\
&= d\left\{\frac{\sum_{j=1}^n (-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{[\bar{S}(\zeta,\xi)S(\zeta,\xi)]^n}\right\}
\end{aligned}$$

$$
\begin{aligned}
&= (-n)[(\bar{S}(\zeta,\xi)S(\zeta,\xi)]^{-(n+1)}\bar{S}_k(\zeta,\xi)\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)\wedge\\
&\quad\wedge(-1)^{k-1+k-1}dS_1\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n\\
&\quad+(-n)[(\bar{S}(\zeta,\xi)S(\zeta,\xi)]^{-(n+1)}S_j(\zeta,\xi)\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)\wedge\\
&\quad\wedge(-1)^{k-1}(-1)^{n-1+j-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots dS_n\wedge d\bar{S}_1\wedge\cdots\wedge\cdots\wedge d\bar{S}_n\\
&\quad+\frac{1}{[\bar{S}(\zeta,\xi)S(\zeta,\xi)]^n}\sum_{j=1}^{n}(-1)^{j-1}\cdot(-1)^{k-1}\cdot\\
&\quad\times(-1)^{n-1+j-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge d\bar{S}_j\wedge\cdots\wedge d\bar{S}_n\\
&= (-n)\frac{\bar{S}_k(\zeta,\xi)\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)dS_1\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n+2}}\\
&\quad+(-n)\frac{\sum_{j=1}^{n}(-1)^{k+n}S_j(\zeta,\xi)\bar{S}_j(\zeta,\xi)dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n+2}}\\
&\quad+n\frac{\sum_{j=1}^{n}(-1)^{k+n}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}\\
&= (-n)\frac{\bar{S}_k(\zeta,\xi)\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)dS_1\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n+2}}\\
&\quad+(-n)\frac{\sum_{j=1}^{n}|S_j(\zeta,\xi)|^2(-1)^{k+n}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n+2}}\\
&\quad+n\frac{\sum_{j=1}^{n}(-1)^{k+n}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}\\
&= (-n)\frac{\bar{S}_k(\zeta,\xi)\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)dS_1\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n+2}}\\
&= (-n)\frac{\bar{S}_k(\zeta,\xi)}{|S(\zeta,\xi)|^2\varphi^{\nu}(\zeta,\xi)}\cdot\\
&\times\frac{\varphi^{\nu}(\zeta,\xi)\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)dS_1\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}\\
&= (-n)\frac{(2\pi i)^n}{(n-1)!}\frac{\bar{S}_k(\zeta,\xi)}{|S(\zeta,\xi)|^2\varphi^{\nu}(\zeta,\xi)}\Omega(\varphi^{\nu},\bar{S},S),
\end{aligned}
\tag{3.3}
$$

where

$$
\Omega(\varphi^{\nu},\bar{S},S)=\frac{(n-1)!}{(2\pi i)^n}\varphi^{\nu}(\zeta,\xi)\frac{w'_{\zeta}(\bar{S}(\zeta,\xi))\wedge w_{\zeta}(S(\zeta,\xi))}{|S(\zeta,\xi)|^{2n}}.
$$

So,

$$
\begin{aligned}
&\int\limits_{\partial D\setminus B(\zeta,\delta)} f(\xi)\frac{\bar{S}_k(\zeta,\xi)}{|S(\zeta,\xi)|^2}\Omega(\varphi^{\nu}(\zeta,\xi)\bar{S}(\zeta,\xi),S(\zeta,\xi))\\
&= -\frac{1}{n}C_n\int\limits_{\partial D\setminus B(\zeta,\delta)}[f(\xi)\varphi^{\nu}(\zeta,\xi)]\cdot\\
&\quad\times d\left[\frac{\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}\right]
\end{aligned}
$$

$$= -\frac{1}{n}C_n \int\limits_{\partial D\setminus B(\zeta,\delta)} d\Big[f(\xi)\varphi^{\nu}(\zeta,\xi)\cdot$$
$$\times \frac{\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}\Bigg]$$
$$+\frac{1}{n}C_n \int\limits_{\partial D\setminus B(\zeta,\delta)} d[f(\xi)\varphi^{\nu}(\zeta,\xi)]\cdot$$
$$\times \frac{\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}. \tag{3.4}$$

By using Stokes' formula to the first term of the end of the above equality chain, we immediately obtain the desired formula in the Lemma. The proof is complete. □

Now we consider the integrals on the right hand side of formula (3.2). Suppose that f is a differentiable function in a neighborhood of the boundary ∂D. On the boundary ∂D, the first-order derivatives of f satisfy the Hölder continuity condition, the exponent $\alpha > 0$. For the first term, the dimension of the integral region is $2n-2$, but the order of a singular point of the integrand is $2n-1$, so this integral is divergent. The second integral exists in the sense of the Cauchy principal value. In relation to the idea of the Hadamard principal value, we have

Definition 3.2 Suppose that D is a relatively compact domain on a Stein manifold M, function f belong to $H_1(\alpha)$ on ∂D. We define the Hadamard principal value

$$\mathrm{FP}\int\limits_{\partial D_\xi} f(\xi)\frac{\bar{S}_k(\zeta,\xi)}{|S(\zeta,\xi)|^2}\Omega(\varphi^{\nu}(\zeta,\xi)\bar{S}(\zeta,\xi),S(\zeta,\xi))$$
$$= \mathrm{PV}\frac{1}{n}C_n\int\limits_{\partial D_\xi} d[f(\xi)\varphi^{\nu}(\zeta,\xi)]\cdot$$
$$\times \frac{\sum_{j=1}^{n}(-1)^{j-1}\bar{S}_j(\zeta,\xi)(-1)^{k-1}dS_1\wedge\cdots\wedge[dS_k]\wedge\cdots\wedge dS_n\wedge d\bar{S}_1\wedge\cdots\wedge[d\bar{S}_j]\wedge\cdots\wedge d\bar{S}_n}{|S(\zeta,\xi)|^{2n}}. \tag{3.5}$$

Since the above definition throws away the divergent part of the higher order singular integral and only keeps its finite part of the integral, we can simplify our computation, and use the result of the Cauchy principal value directly.

Remark When the Stein manifold is just the space C^n, now $S(z,\xi)=\xi-z$. Our definition above reduces to the Hadamard principal value in C^n(refer to Definition 1 of [4]).

Lemma 3.3 (Composite formula for Cauchy type singular integrals on Stein manifolds) *Suppose that $\phi(\eta)\in C^{(1)}(\partial D)$, it can be holomorphically extended to D, then*

$$\int\limits_{\partial D_\xi}\Omega(\varphi^{\nu},\bar{S}(\zeta,\xi),S(\zeta,\xi))\int\limits_{\partial D_\eta}\phi(\eta)\Omega(\varphi^{\mu},\bar{S}(\xi,\eta),S(\xi,\eta))=\frac{1}{4}\phi(\zeta). \tag{3.6}$$

For a proof we refer to [9–11].

Theorem 3.4 (Composite formula for higher order singular integral) *Suppose that D is a relatively compact domain on Stein manifold M, denoted by $U(\partial D)$ a neighborhood of a point on the boundary ∂D. $\phi(\eta)$ is holomorphic in $U(\partial D)$, it can be holomorphically extended into D, then there holds the composite formula*

$$\int_{\partial D_\xi} \Omega(\varphi^\nu(\zeta,\xi),\bar{S}(\zeta,\xi),S(\zeta,\xi)) \int_{\partial D_\eta} \phi(\eta)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^\mu,\bar{S},S)$$
$$= I(\phi(\zeta)) + \frac{1}{4n}\frac{\partial\phi(\zeta)}{\partial\zeta_k}, \tag{3.7}$$

where $I(\phi(\zeta))$ is constituted by integrals in the ordinary sense and in the Cauchy principal value sense (also see formulas (3.38) *and* (3.39) *below).*

If we denote

$$S\phi = 2\int_{\partial D_\xi} \Omega(\varphi^\nu,\bar{S}(\zeta,\xi),S(\zeta,\xi)), \tag{3.8}$$

and

$$S_1\phi = 2\int_{\partial D_\eta} \phi(\eta)\Omega_1(\varphi^\mu,\bar{S}(\xi,\eta),S(\xi,\eta)) = \psi(\xi), \tag{3.9}$$

where

$$\Omega_1(\varphi^\mu,\bar{S},S) = \frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^\mu,\bar{S},S),$$

then the composite formula becomes

$$SS_1\phi = 4\int_{\partial D_\xi} \Omega(\varphi^\nu(\zeta,\xi),\bar{S}(\zeta,\xi),S(\zeta,\xi)) \int_{\partial D_\eta} \phi(\eta)\Omega_1(\varphi^\mu(\xi,\eta),\bar{S}(\xi,\eta),S(\xi,\eta))$$
$$= \frac{1}{n}\frac{\partial\phi(\zeta)}{\partial\zeta_k} + I(\phi(\zeta)) = S(\psi). \tag{3.10}$$

This is an integral-differential equation for which we can obtain a unique solution ϕ under appropriate boundary value conditions.

Proof We will prove the Theorem by localization method. We need to compute the above integrals and the term $\Omega(\varphi^\nu,\bar{S}(z,\eta),S(z,\eta))$ in a local coordinate. For a fixed point $\zeta \in \partial D$, we take $\delta > 0$, $V_{\zeta,\delta} \subset U_j$ is a small coordinate neighborhood of point ζ, $V_{\zeta,\delta} \to B_{\zeta^*,\delta} = \{\eta^* \in C^n : |\eta^* - \zeta^*| < \delta\}$ is a biholomorphic mapping. Assume that $\{U_j\}$ is a local finite cover of M which consist of coordinate neighborhoods, $W_\zeta = S(\zeta,\cdot)$. Denoted by $H_\zeta = W_\zeta(V_{\zeta,\delta} \cap \partial D)$ a hypersurface in $B_{\zeta^*,\delta}$

which passes through the center ζ^*. When a point $z \in V_{\zeta,\delta} \cap D$ tends to ζ sufficiently along a non-tangential direction, there exists a point $z^* \in W_\zeta(V_{\zeta,\delta} \cap D)$ such that $z = W_\zeta^{-1}(z^*)$.

$$\Omega(\varphi^\nu, \bar{S}(z,\eta), S(z,\eta)) = \Omega(\varphi^\nu, \hat{\bar{S}}, \hat{S})(\eta^*) = \frac{(n-1)!}{(2\pi i)^n} \hat{\varphi}^\nu(z^*,\eta^*)\omega' \left(\frac{\hat{\bar{S}}(z^*,\eta^*)}{|\hat{\bar{S}}(z^*,\eta^*)|^2} \right) \wedge (\hat{S}(z^*,\eta^*)), \tag{3.11}$$

where

$$\begin{aligned} \hat{\bar{S}}(z^*,\eta^*) &= \bar{S}(W_\zeta^{-1}(z^*), W_\zeta^{-1}(\eta^*)), \\ \hat{S}(z^*,\eta^*) &= S(W_\zeta^{-1}(z^*), W_\zeta^{-1}(\eta^*)), \\ \hat{\varphi}(z^*,\eta^*) &= \varphi(W_\zeta^{-1}(z^*), W_\zeta^{-1}(\eta^*)). \end{aligned} \tag{3.12}$$

Now we study the relation between the Bochner–Martinelli kernel $\Omega(\varphi^\nu, \hat{\bar{S}}, \hat{S})(\eta^*)$ on the Stein manifold and the kernel $\Omega(\bar{\eta}^* - \bar{z}^*, \eta^* - z^*)$ in C^n.

Since

$$\hat{S}(\cdot,\cdot) : B_{\zeta^*,\delta} \times B_{\zeta^*,\delta} \to \tilde{T}(M \times M),$$

we have

$$\hat{\bar{S}}(\cdot,\cdot) : B_{\zeta^*,\delta} \times B_{\zeta^*,\delta} \to \tilde{T}^*(M \times M). \tag{3.13}$$

Let $\{u_j\}$, $\{\bar{u}_j\}$ be local coordinates of $\hat{S}$, $\hat{\bar{S}}$. $u_j(z^*,\eta^*)$ are holomorphic functions defined in the convex region $B_{\zeta^*,\delta} \times B_{\zeta^*,\delta}$, $u(z^*,z^*) = 0$. By division theorem we can obtain the expression

$$u_j(z^*,\eta^*) = \sum_{k=1}^{n} \gamma_{jk}(z^*,\eta^*)(\eta_k^* - z_k^*), \tag{3.14}$$

where $\gamma_{jk}(z^*,\eta^*)$ is holomorphic for z^* and η^*. In addition,

$$u(0,\eta^*) = \hat{S}(0,\eta^*) = S(\zeta, W_\zeta^{-1}(\eta^*)) = W_\zeta(W_\zeta^{-1}(\eta^*)) = \eta^* \quad (\text{here } S(\zeta,\cdot) = W_\zeta), \tag{3.15}$$

when (z^*,η^*) is in a sufficiently small neighborhood of (ζ^*,ζ^*). We might as well suppose this neighborhood is $B_{\zeta^*,\delta} \times B_{\zeta^*,\delta}$,

$$det(\gamma_{jk}(z^*,\eta^*)) \neq 0. \tag{3.16}$$

By division theorem again, γ_{jk} can be expressed as

$$\gamma_{jk}(z^*, \eta^*) = \sum_{l=1}^{n} \gamma_{jkl}(z^*, \eta^*) z_l^* + \delta_{jk}, \quad (z^*, \eta^*) \in B_{\zeta^*,\delta} \times B_{\zeta^*,\delta}. \tag{3.17}$$

where $\delta_{jk} = \gamma_{jk}(0, \eta^*)$.

By formula (3.13)–(3.16) we know kernel $\Omega(\hat{\varphi}^\nu, \hat{\bar{S}}, \hat{S})(\eta^*)$ can be expressed as

$$\Omega(\hat{\varphi}^\nu, \hat{\bar{S}}, \hat{S})(\eta^*) = \hat{\varphi}^\nu(z^*, \eta^*) B(z^*, \eta^*) + \hat{\varphi}^\nu(z^*, \eta^*) A(z^*, \eta^*), \tag{3.18}$$

where the exterior differential formula B does not include $d\gamma_{jk}$, but A includes $d\gamma_{jk}$. From reference [2,7,12,13], we have

$$|A(z^*, \eta^*)) = O(|\eta^* - z^*|^{2-2n}), \tag{3.19}$$

$$B(z^*, \eta^*) = \frac{(n-1)!}{(2\pi i)^n} \frac{det(\bar{\Gamma}_t \Gamma)\omega'(\bar{\eta}^* - \bar{z}^*) \wedge \omega(\eta^* - z^*)}{[\sum_{j,k,l} \gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^* - \bar{z}_l^*)(\eta_j^* - z_j^*)]^n}, \tag{3.20}$$

where $\Gamma = \gamma_{jk}(z^*, \eta^*)$, Γ_t is a transposition of Γ.

Moreover, we know the difference between $B(z^*, \eta^*)$ and the Bochner–Martinelli kernel $\Omega(\bar{\eta}^* - \bar{z}^*, \eta^*, -z^*)$ in C^n is

$$|B(z^*, \eta^*) - \Omega(\bar{\eta}^* - \bar{z}^*, \eta^* - z^*)| = O(|\eta^* - z^*|^{2-2n}). \tag{3.21}$$

In addition, we have estimate

$$|\hat{\varphi}^\nu(z^*, \eta^*) - 1| = |\hat{\varphi}^\nu(z^*, \eta^*) - \hat{\varphi}^\nu(z^*, z^*)| = O(|\eta^* - z^*|). \tag{3.22}$$

When z^* tends to ξ^*, we have the following estimates similar to formula (3.21) and (3.22)

$$|B(\xi^*, \eta^*) - \Omega(\bar{\eta}^* - \bar{\xi}^*, \eta^* - \xi^*)| = O(|\eta^* - \xi^*|^{2-2n}) \tag{3.23}$$

and

$$|\hat{\varphi}^\nu(\xi^*, \eta^*) - 1| = O(|\eta^* - \xi^*|). \tag{3.24}$$

For the kernel

$$\frac{\bar{S}_k(z, \eta)}{|S(z, \eta)|^2} \Omega(\varphi^\mu, \bar{S}(z, \eta), S(z, \eta)),$$

based on the previous knowledge, we have the following relations and estimations

$$\frac{\bar{S}_k(z,\eta)}{|S(z,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(z,\eta),S(z,\eta)) = \frac{\hat{\bar{S}}_k}{|\hat{S}|^2}\Omega(\hat{\varphi}^\mu,\hat{\bar{S}},S)(\eta^*)$$
$$= \hat{\varphi}^\mu(z^*,\eta^*)\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{|\sum_{j.k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-\bar{z}_j^*)|}[B(z^*,\eta^*)+A(z^*,\eta^*)], \tag{3.25}$$
$$\left|\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{|\sum_{j.k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-z_j^*)|}A(z^*,\eta^*)\right| = O(|\eta^*-z^*|^{1-2n}), \tag{3.26}$$
$$\left|\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{|\sum_{j.k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-\bar{z}_j^*)|}B(z^*,\eta^*)-\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\right|$$
$$= O(|\eta^*-z^*|^{1-2n}), \tag{3.27}$$
$$|\hat{\varphi}^\mu(z^*,\eta^*)-1| = |\hat{\varphi}^\mu(z^*,\eta^*)-\hat{\varphi}^\mu(z^*,z^*)| = O(|\eta^*-z^*|). \tag{3.28}$$

When z^* tends to ξ^*, under the biholomorphic mapping $V_{\zeta,\delta}\to B_{\zeta^*,\delta}$, we also have the estimates in the local coordinate neighborhood of $B_{\zeta^*,\delta}\times B_{\zeta^*,\delta}$

$$\left|\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{\xi}_j^*)}{|\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{\xi}_l^*)(\eta_j^*-\xi_j^*)|}B(\xi,\eta^*)-\frac{\bar{\eta}_k^*-\bar{\xi}_k^*}{|\eta^*-\xi|^2}\Omega(\bar{\eta}^*-\bar{\xi}^*,\eta^*-\xi^*)\right|$$
$$= O(|\eta^*-\xi^*|^{1-2n}), \tag{3.29}$$
$$|\hat{\varphi}^\mu(\xi^*,\eta^*)-1| = O(|\eta^*-\xi^*|). \tag{3.30}$$

Let

$$F(z) = \int\limits_{\partial D_\xi}\Omega(\varphi^\nu,\bar{S}(\zeta,\xi),S(\zeta,\xi))\int\limits_{\partial D_\eta}\phi(\eta)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\xi,\eta),S(\xi,\eta)). \tag{3.31}$$

Then

$$F(z) = \left\{\int\limits_{\xi\in\partial D\setminus V_{\zeta,\delta\cap\partial D}}\Omega(\varphi^\nu,\bar{S}(\zeta,\xi),S(\zeta,\xi)) + \int\limits_{\xi^*\in H_\zeta}[\hat{\varphi}^\nu(z^*,\xi^*)A(z^*,\xi^*)\right.$$
$$+\hat{\varphi}^\nu(z^*,\xi^*)(B(z^*,\xi^*)-\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*))$$
$$\left.+(\hat{\varphi}^\nu(z^*,\xi^*)-1)\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*)+\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*)]\right\}$$
$$\times\left\{\int\limits_{\eta\in\partial D\setminus V_{\xi,\varepsilon}\cap\partial D}\phi(\eta)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\cdot\Omega(\varphi^\mu,\bar{S}(\xi,\eta),S(\xi,\eta))\right.$$

$$+\int_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\left[\hat{\varphi}^\mu(z^*,\eta^*)\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-z_j^*)}A(z^*,\eta^*)+\hat{\varphi}^\mu(z^*,\eta^*)\right.$$
$$\times\left(\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-z_j^*)}B(z^*,\eta^*)-\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\right)$$
$$+(\hat{\varphi}^\mu(z^*,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)$$
$$\left.\left.+\ \frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\right]\right\},\tag{3.32}$$

where

$$\hat{\phi}(\eta^*)=\phi(W_\zeta^{-1}(\eta^*)).\tag{3.33}$$

When we use the previously recalled estimates, it is easy to find that the first four terms of both the two brackets in the right hand side of the above expression are proper integrals. The fifth term in the first bracket is a Bochner–Martinelli integral in C^n, and the fifth term in the second bracket is a higher order Bochner–Martinelli integral.

For simplicity, we separate the second integral in the first bracket of formula (3.32) into two parts, one is

$$\int_{\xi^*\in H_\zeta}\Phi(z^*,\xi^*)=\int_{\xi^*\in H_\zeta}[\hat{\varphi}^\nu(z^*,\xi^*)A(z^*,\xi^*)+\hat{\varphi}^\nu(z^*,\xi^*)(B(z^*,\xi^*)$$
$$-\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*))+(\hat{\varphi}^\nu(z^*,\xi^*)-1)\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*)],\tag{3.34}$$

the other is

$$\int_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*),$$

we also separate the second integral in the second bracket of formula (3.32) into two parts, one is

$$\int_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(z^*,\eta^*)$$
$$=\int_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\left[\hat{\varphi}^\mu(z^*,\eta^*)\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-z_j^*)}A(z^*,\eta^*)+\hat{\varphi}^\mu(z^*,\eta^*)\right.$$

$$\times\left(\frac{\sum_{j=1}^{n}\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-z_j^*)}B(z^*,\eta^*)-\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\right)$$
$$+(\hat{\varphi}^{\mu}(z^*,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\Bigg], \tag{3.35}$$

the other is

$$\int\limits_{\eta^*\in H_\xi}\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*),$$

then formula (3.32) becomes

$$\begin{aligned}
F(z)=&\Bigg\{\int\limits_{\xi\in\partial D\setminus V_{\zeta,\delta}\cap\partial D}\Omega(\varphi^{\nu},\bar{S}(\zeta,\xi),S(\zeta,\xi))+\int\limits_{\xi^*\in H_\zeta}\Phi(z^*,\xi^*)\\
&+\int\limits_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*)\Bigg\}\Bigg\{\int\limits_{\eta\in\partial D\setminus V_{\xi,\varepsilon}\cap\partial D}\phi(\eta)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^{\mu},\bar{S}(\xi,\eta),S(\xi,\eta))\\
&+\int\limits_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(z^*,\eta^*)+\int\limits_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\Bigg\}\\
=&\int\limits_{\xi\in\partial D\setminus V_{\zeta,\delta}\cap\partial D}\Omega(\varphi^{\nu},\bar{S}(\zeta,\xi),S(\zeta,\xi))\int\limits_{\eta\in\partial D\setminus V_{\xi,\varepsilon}\cap\partial D}\phi(\eta)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^{\mu},\bar{S}(\xi,\eta),S(\xi,\eta))\\
&+\int\limits_{\xi\in\partial D\setminus V_{\zeta,\delta}\cap\partial D}\Omega(\varphi^{\nu},\bar{S}(\zeta,\xi),S(\zeta,\xi))\int\limits_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(z^*,\eta^*)\\
&+\int\limits_{\xi\in\partial D\setminus V_{\zeta,\delta}\cap\partial D}\Omega(\varphi^{\nu},\bar{S}(\zeta,\xi),S(\zeta,\xi))\int\limits_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\\
&+\int\limits_{\xi^*\in H_\zeta}\Phi(z^*,\xi^*)\int\limits_{\eta\in\partial D\setminus V_{\xi,\varepsilon}\cap\partial D}\phi(\eta)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^{\mu},\bar{S}(\xi,\eta),S(\xi,\eta))\\
&+\int\limits_{\xi^*\in H_\zeta}\Phi(z^*,\xi^*)\int\limits_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(z^*,\eta^*)\\
&+\int\limits_{\xi^*\in H_\zeta}\Phi(z^*,\xi^*)\int\limits_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\\
&+\int\limits_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*)\int\limits_{\eta\in\partial D\setminus V_{\xi,\varepsilon}\cap\partial D}\hat{\phi}(\eta^*)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^{\mu},\bar{S}(\xi,\eta),S(\xi,\eta))\\
&+\int\limits_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*)\int\limits_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(z^*,\eta^*)
\end{aligned}$$

$$+\int_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-\bar{z}^*,\xi^*-z^*)\int_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*). \tag{3.36}$$

Suppose that the hypersurface $H_\zeta = W_\zeta(V_{\zeta,\delta}\cap\partial D)$ passes through the origin O, now we set $\zeta^*=0$. When $z^*\to 0$, recalling the definition of the Hadamard principal value of higher order singular integrals in C^n (Definition 1 in [4]) and using some of the previous estimates, we have

$$\begin{aligned}
\Phi(\zeta) = &\int_{\xi\in\partial D\backslash V_{\zeta,\delta}\cap\partial D}\Omega(\varphi^\nu,\bar{S}(\zeta,\xi),S(\zeta,\xi))\int_{\eta\in\partial D\backslash V_{\xi,\varepsilon}\cap\partial D}\phi(\eta)\cdot\\
&\cdot\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\xi,\eta),S(\xi,\eta))\\
&+\int_{\xi\in\partial D\backslash V_{\zeta,\delta}\cap\partial D}\Omega(\varphi^\nu,\bar{S}(\zeta,\xi),S(\zeta,\xi))\int_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(0,\eta^*)\\
&+\int_{\xi\in\partial D\backslash V_{\zeta,\delta}\cap\partial D}\Omega(\varphi^\nu,\bar{S}(\zeta,\xi),S(\zeta,\xi))\mathrm{PV}\int_{\eta^*\in H_\xi}\frac{1}{n}\frac{\partial\hat{\phi}(\eta^*)}{\partial\eta_k^*}\Omega(\bar{\eta}^*-0,\eta^*-0)\\
&+\int_{\xi^*\in H_\zeta}\Phi(0,\xi^*)\int_{\eta\in\partial D\backslash V_{\xi,\varepsilon}\cap\partial D}\phi(\eta)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\xi,\eta),S(\xi,\eta))\\
&+\int_{\xi^*\in H_\zeta}\Phi(0,\xi^*)\int_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(0,\eta^*)\\
&+\int_{\xi^*\in H_\zeta}\Phi(0,\xi^*)\mathrm{PV}\int_{\eta^*\in H_\xi}\frac{\partial\hat{\phi}(\eta^*)}{\partial\eta_k^*}\Omega(\bar{\eta}^*-0,\eta^*-0)\\
&+\int_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-0,\xi^*-0)\int_{\eta\in\partial D\backslash V_{\xi,\varepsilon}\cap\partial D}\hat{\phi}(\eta^*)\frac{\bar{S}_k(\xi,\eta)}{|S(\xi,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\xi,\eta),S(\xi,\eta))\\
&+\int_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-0,\xi^*-0)\int_{\eta^*\in H_\xi}\hat{\phi}(\eta^*)\Psi(0,\eta^*)\\
&+\int_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-0,\xi^*-0)\mathrm{PV}\int_{\eta^*\in H_\xi}\frac{1}{n}\frac{\partial\hat{\phi}(\eta^*)}{\partial\eta_k^*}\Omega(\bar{\eta}^*-0,\eta^*-0).
\end{aligned}\tag{3.37}$$

By the previous estimates, we see that the first eight integrals in the right hand side of the above formula are proper integrals and they exist in the sense of the Cauchy principal value. For simplicity, denoted by $I(\phi(\zeta))$ the sum of the first eight integrals, then formula (3.37) becomes

$$\Phi(\zeta) = I(\phi(\zeta)) + \int_{\xi^*\in H_\zeta}\Omega(\bar{\xi}^*-0,\xi^*-0)\mathrm{PV}\int_{\eta^*\in H_\xi}\frac{1}{n}\frac{\partial\hat{\phi}(\eta^*)}{\partial\eta_k^*}\Omega(\bar{\eta}^*-0,\eta^*-0). \tag{3.38}$$

For the second term in the expression (3.38), applying the composite formula for singular integral with Bochner–Martinelli kernel in C^n (refer to 4, 6 and 8), we have

$$\int_{\xi^*\in H_\zeta} \Omega(\bar{\xi}^*-0,\xi^*-0)\mathrm{PV}\int_{\eta^*\in H_\xi} \frac{1}{n}\frac{\partial\hat{\phi}(\eta^*)}{\partial\eta_k^*}\Omega(\bar{\eta}^*-0,\eta^*-0)=\frac{1}{4n}\frac{\partial\phi(\zeta)}{\partial\zeta_k}. \tag{3.39}$$

The proof is complete. □

Remark In particular, when the Stein manifold M is just the space C^n, we have that the section $S(z,\xi)=\xi-z$, and the composite formula (3.7) on the Stein manifold M becomes the composite formula in the space C^n which is

$$\int_{\partial D_\xi} K(\zeta,\xi)\int_{\partial D_\eta} \phi(\eta)\frac{\xi_k-\eta_k}{|\xi-\eta|^2}K(\xi,\eta)=\frac{1}{4n}\frac{\partial\phi(\zeta)}{\partial\zeta_k}. \tag{3.40}$$

Comparing the composite formula (3.7) on the Stein manifold with the composite formula (3.40) in C^n, we find that there is an extra term $I(\phi(\zeta))$ for the Stein manifold case that is caused by the difference of the section $S(z,\xi)$ of the Stein manifold and the section $\xi-z$ of C^n. In local coordinate, the relationship between the coordinate expression of $S(z,\xi)$ and $\xi_k^*-z_k^*$ is $u_j(z^*,\xi^*)=\sum_{k=1}^n\gamma_{jk}(z^*,\xi^*)(\xi_k^*-z_k^*)$, where $\gamma_{jk}(z^*,\xi^*)$ is holomorphic for z^* and ξ^*, this is a nonlinear relationship.

Theorem 3.5 (Plemelj formulas for higher order singular integrals) *Suppose that D is a relatively compact domain on the Stein manifold M. f is differentiable in a neighborhood of the boundary ∂D and belong to $H_1(\alpha)$ on $\bar{D}$. For the higher order Bochner–Martinelli type integral, we have*

$$F(z)=\int_{\partial D_\eta} f(\eta)\frac{\bar{S}_k(z,\eta)}{|S(z,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(z,\eta),S(z,\eta))\quad z\in M\backslash\partial D, \tag{3.41}$$

when z tends to $\zeta\in\partial D$ from the inner part or the outer part of D, we have the Plemelj formulas respectively as follows

$$F_i(\zeta)=\mathrm{FP}\int_{\eta\in\partial D} f(\eta)\frac{\bar{S}_k(\zeta,\eta)}{|S(\zeta,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\zeta,\eta),S(\zeta,\eta))$$
$$+\frac{1}{2}f(\zeta)[\hat{\varphi}^\mu(0,\eta^*)-1]\frac{\bar{\eta}^*-\bar{0}}{|\eta^*-0|^2}+\frac{1}{2n}\left[\frac{\partial f(\zeta)}{\partial\zeta_k}+(-1)^n\frac{\partial f(\zeta)}{\partial\zeta_1}\right] \tag{3.42}$$

or

$$F_e(\zeta) = \mathrm{FP}\int\limits_{\eta\in\partial D} f(\eta)\frac{\bar{S}_k(\zeta,\eta)}{|S(\zeta,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\zeta,\eta),S(\zeta,\eta))$$
$$-\frac{1}{2}f(\zeta)[\hat{\varphi}^\mu(0,\eta^*)-1]\frac{\bar{\eta}^*-\bar{0}}{|\eta^*-0|^2}-\frac{1}{2n}\left[\frac{\partial f(\zeta)}{\partial\zeta_k}+(-1)^n\frac{\partial f(\zeta)}{\partial\zeta_1}\right] \tag{3.43}$$

Proof We select a small neighbourhood of $\zeta\in\partial D$ which is $V_{\zeta,\delta}$, and take a biholomorphic mapping $W_\zeta = S(\zeta,\cdot): V_{\zeta,\delta}\to B_{\zeta^*,\delta} = \{\eta^*\in C^n : |\eta^*-\zeta^*|<\delta\}$, here ζ^* is the center of $B_{\zeta^*,\delta}$. Denoted by $H_\zeta = W_\zeta(V_{\zeta,\delta}\cap\partial D)$ a hypersurface in $B_{\zeta^*,\delta}$ which passes through the center of $B_{\zeta^*,\delta}$. When $z\in V_{\zeta,\delta}\cap D$ is sufficiently close to ζ, there is a point $z^*\in W_\zeta(V_{\zeta,\delta}\cap D)$ such that $z = W_\zeta^{-1}(z^*)$. By the proof of Theorem 3.4 we have

$$F(z) = \int\limits_{\partial D_\eta} f(\eta)\frac{\bar{S}_k(z,\eta)}{|S(z,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(z,\eta),S(z,\eta))$$
$$= \int\limits_{\eta\in\partial D\setminus V_{\zeta,\delta}\cap\partial D} f(\eta)\frac{\bar{S}_k(\zeta,\eta)}{|S(\zeta,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\zeta,\eta),S(\zeta,\eta))$$
$$+\int\limits_{\eta^*\in H_\zeta}\hat{f}(\eta^*)\Bigg[\hat{\varphi}^\mu(z^*,\eta^*)\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-z_j^*)}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-z_j^*)}A(z^*,\eta^*)+\hat{\varphi}^\mu(z^*,\eta^*)$$
$$\times\left(\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{z}_j^*)}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{z}_l^*)(\eta_j^*-z_j^*)}B(z^*,\eta^*)-\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\right)$$
$$+(\hat{\varphi}^\mu(z^*,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)$$
$$+\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)\Bigg]. \tag{3.44}$$

From the estimate formulas (3.26), (3.27), (3.28), (3.29) and (3.30), we find the first and the second terms of the second integral in the above expression are proper integrals. Following we consider the third and the fourth terms of the second integral.

The third term is a singular integral with Bochner–Martinelli kernel in C^n. For simplicity we suppose the center of $B_{\zeta^*,\delta}$ is at the origin O, so $\zeta^*=0$. When z^* tends to 0, using the Plemelj formula in C^n, we have

$$\lim_{z\to 0^+}\int\limits_{\eta^*\in H_\zeta}\hat{f}(\eta^*)(\hat{\varphi}^\mu(z^*,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)$$
$$=\mathrm{PV}\int\limits_{\eta^*\in H_\zeta}\hat{f}(\eta^*)(\hat{\varphi}^\mu(0,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{0}}{|\eta^*-0|^2}\Omega(\bar{\eta}^*-\bar{0},\eta^*-0)$$
$$+\frac{1}{2}\hat{f}(0)(\hat{\varphi}^\mu(0,\eta^*)-1)\frac{\bar{\eta}^*-\bar{0}}{|\eta^*-0|^2} \tag{3.45}$$

or

$$\lim_{z\to 0^-}\int_{\eta^*\in H_\zeta}\hat{f}(\eta^*)(\hat{\varphi}^\mu(z^*,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)$$
$$=\mathrm{PV}\int_{\eta^*\in H_\zeta}\hat{f}(\eta^*)(\hat{\varphi}^\mu(0,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{0}}{|\eta^*-0|^2}\Omega(\bar{\eta}^*-\bar{0},\eta^*-0)$$
$$-\frac{1}{2}\hat{f}(0)(\hat{\varphi^\mu}(0,\eta^*)-1)\frac{\bar{\eta}^*-\bar{0}}{|\eta^*-0|^2}, \tag{3.46}$$

where $\hat{f}(0)=f(\zeta)$.

The last term in formula (3.44) is a higher order singular integral in C^n. According to Theorem 1 of [4], we have

$$\lim_{z\to 0^+}\int_{\eta^*\in H_\zeta}\hat{f}(\eta^*)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)$$
$$=\mathrm{FP}\int_{\eta^*\in H_\zeta}\hat{f}(\eta^*)\frac{\bar{\eta}_k^*-0}{|\eta^*-0|^2}\Omega(\bar{\eta}^*-\bar{0},\eta^*-0)$$
$$+\frac{1}{2n}\left[\frac{\partial f(\zeta)}{\partial\zeta_k}+(-1)^n\frac{\partial f(\zeta)}{\partial\zeta_1}\right], \tag{3.47}$$

or

$$\lim_{z\to 0^-}\int_{\eta^*\in H_\zeta}\hat{f}(\eta^*)\frac{\bar{\eta}_k^*-\bar{z}_k^*}{|\eta^*-z^*|^2}\Omega(\bar{\eta}^*-\bar{z}^*,\eta^*-z^*)$$
$$=\mathrm{FP}\int_{\eta^*\in H_\zeta}\hat{f}(\eta^*)\frac{\bar{\eta}_k^*-0}{|\eta^*-0|^2}\Omega(\bar{\eta}^*-\bar{0},\eta^*-0)-\frac{1}{2n}\left[\frac{\partial f(\zeta)}{\partial\zeta_k}+(-1)^n\frac{\partial f(\zeta)}{\partial\zeta_1}\right]. \tag{3.48}$$

Let Hadamard principal value

$$\mathrm{FP}\int_{\partial D_\eta}f(\eta)\frac{\bar{S}_k(\zeta,\eta)}{|S(\zeta,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\zeta,\eta),S(\zeta,\eta))$$
$$=\int_{\eta\in\partial D\setminus V_{\zeta,\delta}\cap\partial D}f(\eta)\frac{\bar{S}_k(\zeta,\eta)}{|S(\zeta,\eta)|^2}\Omega(\varphi^\mu,\bar{S}(\zeta,\eta),S(\zeta,\eta))$$
$$+\int_{\eta^*\in H_\zeta}\hat{f}(\eta^*)\left[\hat{\varphi}^\mu(0,\eta^*)\frac{\sum_{j=1}^n\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{0})}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{0})(\eta_j^*-0)}A(0,\eta)+\hat{\varphi}^\mu(0,\eta^*)\right.$$

$$\times\left(\frac{\sum_{j=1}^{n}\bar{\gamma}_{jk}(\bar{\eta}_j^*-\bar{0})}{\sum_{j,k,l}\gamma_{jk}\bar{\gamma}_{kl}(\bar{\eta}_l^*-\bar{0})(\eta_j^*-0)}B(0,\eta^*)-\frac{\bar{\eta}_k^*-\bar{0}}{|\eta^*-0|^2}\Omega(\bar{\eta}^*-\bar{0},\eta^*-0)\right)$$
$$+\mathrm{PV}\int\limits_{\eta^*\in H_\zeta}\hat{f}(\eta^*)(\hat{\varphi}^{\mu}(0,\eta^*)-1)\frac{\bar{\eta}_k^*-\bar{0}}{|\eta^*-0|^2}\Omega(\bar{\eta}^*-\bar{0},\eta^*-0)$$
$$\left.+\,\mathrm{FP}\int\limits_{\eta^*\in \mathrm{H}_\zeta}\hat{f}(\eta^*)\frac{\bar{\eta}_k^*-\bar{0}}{|\eta^*-0|^2}\Omega(\bar{\eta}^*-\bar{0},\eta^*-0)\right], \tag{3.49}$$

when we apply the results of formula (3.45), (3.46), (3.47) and (3.48) to formula (3.44), we have the desired formula (3.42) and (3.43). □

4 Higher Order Singular Integral-differential Equation on Stein Manifolds

Suppose that $\phi(\eta)$ is holomorphic in a neighborhood of ∂D, then the composite formula (3.7) holds. We can accordingly solve the higher order singular integral equations with the Bochner–Martinelli kernel as follows.

Consider the higher order singular integral equation

$$aS+bS_1\phi=\psi, \tag{4.1}$$

where a, b are complex value constants, S, S_1 are given by the formulas (3.8) and (3.9). By Definition 3.2, Eq. (4.1) is a partial differential singular integral equation.

Applying the operator

$$M=aI-bS,\quad I\phi=\phi \tag{4.2}$$

to both sides of the characteristic equation, we have

$$a^2S\phi+abS_1\phi-abSS\phi-b^2SS_1\phi=(aI-bS)\psi,$$

that is

$$a(aS\phi+bS_1\phi)-abSS\phi-b^2SS_1\phi=(aI-bS)\psi.$$

So

$$\begin{aligned}a\psi-abSS\phi-b^2SS_1\phi&=a\psi-bS\psi,\\ aSS\phi+bSS_1\phi&=S\psi.\end{aligned} \tag{4.3}$$

In Theorem 3.4 (see formula 3.10), we have the results

$$SS\phi=\phi,\quad SS_1\phi=I(\phi(\zeta))+\frac{1}{n}\frac{\partial\phi(\zeta)}{\partial\zeta_k},$$

applying them to (4.3), we have

$$a\phi + bI(\phi(\zeta)) + \frac{b}{n}\frac{\partial \phi(\zeta)}{\partial \zeta_k} = S\psi, \tag{4.4}$$

where $I(\phi(\zeta))$ consists of the proper or the Cauchy principal value integrals.

Under certain boundary value conditions we can solve the integral-differential equation (4.4) for a unique solution of function ϕ (refer to [13]).

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