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Quasihyperbolic Distance in Punctured Planes

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Abstract We give an explicit formula of the quasihyperbolic distance from a point to a line in the once punctured plane and prove the geodesic is orthogonal to the line. By this result, we give an affirmative answer to the open problem in the case of twice and thrice punctured planes raised by Klén and generalize their estimates. We also construct an example to show that the cosine inequality does not hold in twice or thrice punctured planes.

Keywords Quasihyperbolic distance · Quasihyperbolic geodesic · Cosine inequality

Mathematics Subject Classification (2000) Primary 30C65; Secondary 51M25

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1 Introduction

Hyperbolic type metrics play an important role in the geometric theory of functions. In the case of planar simply connected domains of hyperbolic type, we can easily choose the hyperbolic metric to be a hyperbolic type metric, since the Riemann mapping theorem holds and the hyperbolic metric is a conformally invariant metric [1]. For multiply connected domains the situation is more complicated. There does not exist a conformal mapping between two multiply connected domains with the same connectivity. Some multiply connected domains such as a twice punctured plane do not have a hyperbolic metric [6]. In a higher dimensional domains, there does not always exist a hyperbolic metric by Liouville theorem [14]. Gehring and Palka [5] introduced the quasihyperbolic metric which is a metric of hyperbolic type and adapt to a general domain. The boundary of a proper domain Ω is denoted by $\partial\Omega$. Let $d(z, \partial\Omega)$ represent the Euclidean distance between z and $\partial\Omega$. The quasihyperbolic length of a rectifiable curve $J \subset \Omega$ is defined as follows

$$\ell_{k_\Omega}(J) = \int_J w(z)|dz|,$$

where $w : \Omega \to R_+$ is given by $w(z) = \frac{1}{d(z,\partial\Omega)}$. Furthermore, the quasihyperbolic distance between z_1 and z_2 in Ω is defined by

$$k_\Omega(z_1, z_2) = \inf_J \ell_{k_\Omega}(J),$$

where the infimum is taken over all rectifiable curves J in Ω connecting z_1 and z_2. It is clear that if Ω' and Ω are proper domains with $\Omega' \subset \Omega, z_1, z_2 \in \Omega'$ then $k_{\Omega'}(z_1, z_2) \geq k_\Omega(z_1, z_2)$ [9].

Given $z_1, z_2 \in \Omega$, let $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = z_1, \gamma(1) = z_2$ be a quasihyperbolic length minimizing curve such that

$$k_\Omega(z_1, z_2) = \ell_{k_\Omega}(\gamma|[0, t]) + \ell_{k_\Omega}(\gamma|[t, 1]),$$

for all $t \in [0, 1]$. Then γ is called a quasihyperbolic geodesic joining z_1 and z_2 in Ω and denoted by $\gamma : z_1 \curvearrowright z_2$.

The real axis and complex plane are denoted by R, R^2, respectively. Gehring and Osgood [4] proved that there always exists a quasihyperbolic geodesic γ with z_1 and z_2 as its end points. However, very little is known about the structure of a quasihyperbolic geodesics when Ω is given. Martin and Osgood [12] proved that the quasihyperbolic geodesics are logarithmic spirals in $G_1 = R^2\backslash\{o\}$ and the quasihyperbolic distance between two points $z_1, z_2 \in G_1$ is given by

$$k_{G_1}(z_1, z_2) = \sqrt{\alpha^2 + \log^2 \frac{|z_1|}{|z_2|}}, \tag{1.1}$$

where $\alpha = \angle(z_1, o, z_2) \in [0, \pi]$. Moreover, Martin [13] concluded that quasihyperbolic geodesics are Lipschitz continuous with first derivatives.

Väisälä [16] posed and proved three conjectures as follows

Theorem A *Let $z_1, z_2 \in G$ and $G \subset R^2$ be a domain, then*

(1) **Uniqueness conjecture:** *There is a universal constant $c_u > 0$ such that if $a, b \in G$ and $k_G(a, b) < c_u$ then there is only one quasihyperbolic geodesic $\gamma : a \curvearrowright b$. The conjecture holds for G with $c_u = 2$.*
(2) **Prolongation conjecture:** *There is a universal constant $c_p > 0$ such that if $\gamma : a \curvearrowright b$ is a quasihyperbolic geodesic with $\ell_{k_G}(\gamma) = k_G(a, b) < c_p$ then there is a quasihyperbolic geodesic $\gamma_1 : a \curvearrowright b_1$ such that $\gamma \subset \gamma_1$ and $\ell(\gamma_1) = c_p$. The conjecture holds for G with $c_p = 2$.*
(3) **Convexity conjecture:** *There is a universal constant $c_C > 0$ such that the quasihyperbolic ball $B_{k_G}(a, r)$ is strictly convex for all $r < c_C$. The conjecture holds for G with the sharp constant $c_C = 1$.*

Martio and Väisälä [11] proved that all convex domains satisfy the above three conjectures without any restrictions to the quasihyperbolic distance. Klén [10] showed the $c_C \le 1$ in G_1. Lindén [7] studied the quasihyperbolic geodesic behaviors in angular domains. For other topics about hyperbolic type metrics and open problems see [15,17–19].

Recently, Klén [9] studied the quasihyperbolic length of a contour in a punctured plane and proved

Theorem B *Let $\widetilde{\gamma} \subset R^2\backslash\{-1, 1\}$ be a closed rectifiable curve enclosing $\{-1, 1\}$. Then*

$$\ell_{k_{R^2\backslash\{-1,1\}}}(\widetilde{\gamma}) \ge (\pi - \arctan h)\sqrt{1 + \frac{1}{h^2}} + \frac{3\pi}{2},$$

where h is the solution of the equation $d(\widetilde{\gamma}, \{-1, 1\}) = \sqrt{1 + h^2}e^{\frac{\arctan h - \pi}{h}}$.

Furthermore, Klén [9] raised an open problem as follows.

Open problem C *Let $z_1, z_2, z_3, \ldots, z_m \in R^2$ and $\widetilde{\gamma}$ be a simple and closed curve that encloses the points $z_1, z_2, \ldots, z_m$. Find a lower bound for $\ell_{k_G}(\widetilde{\gamma})$, where $G = R^2\backslash\{z_1, z_2, \ldots, z_m\}$.*

In order to study this problem, we first give an explicit formula of the quasihyperbolic distance from an arbitrary point to an arbitrary line in $G_1 = R^2\backslash\{z_0\}$ and prove the quasihyperbolic geodesic γ from the point to the line is orthogonal to the line (see Lemma 2.2). Using this result, we give an affirmative answer to the above open problem C in the case of $R^2\backslash\{z_1, z_2\}$ and $R^2\backslash\{z_1, z_2, z_3\}$.

An arbitrary twice punctured domain can be normalized by $G_2 = R^2\backslash\{-r, r\}, r > 0$. We estimate the lower bound of $\widetilde{\gamma}$ in G_2 enclosing $\{-r, r\}$ and generalize Theorem B proved by Klén.

Theorem 1.1 *Let $\widetilde{\gamma} \subset G_2$ be a closed rectifiable curve enclosing $\{-r, r\}$ and $d(\widetilde{\gamma}, \{-r, r\})$ represent the shortest Euclidean distance of $\widetilde{\gamma}$ to the boundary ∂G_2. Then*

$$\ell_{k_{G_2}}(\widetilde{\gamma}) \geq \left(\pi - \arctan \frac{y}{r}\right)\sqrt{1 + \frac{r^2}{y^2}} + \frac{3}{2}\pi, \tag{1.2}$$

where y satisfies that

$$d(\widetilde{\gamma}, \{-r, r\}) = \sqrt{y^2 + r^2}\, e^{\frac{r}{y}\left(\arctan \frac{y}{r} - \pi\right)}.$$

and $d(\widetilde{\gamma}, \{-r, r\})$ increases in y for a fixed r.

Theorem B is Theorem 1.1 in the case $r = 1$. It's easy to know that the estimate of (1.2) tends to 2π as $d \to +\infty$. Especially, the domain G_2 degenerates to the domain $G_1 = R^2 \backslash \{o\}$ as $r \to 0^+$ and it is easy to get that $\ell_{k_{G_1}}(\gamma) = 2\pi$, so the estimate of (1.2) is asymptotically sharp. Moreover, we build a bridge between G_2 and G_1 by introducing the parameter r.

Furthermore, we give some estimates about the lower bound of the quasihyperbolic length of $\widetilde{\gamma}$ in $R^2 \backslash \{z_1, z_2, z_3\}$ enclosing $\{z_1, z_2, z_3\}$. A thrice punctured plane can be normalized by $G_3 = R^2 \backslash \{-2r_2, 0, 2r_1\}, r_1 > 0, r_2 > 0$ or $G_3' = R^2 \backslash \{r, re^{i2\alpha_1}, re^{i2(\alpha_1+\alpha_2)}\}, r > 0, \alpha_1 > 0, \alpha_2 > 0$ and $\alpha_1 + \alpha_2 < \pi$. If the thrice punctured plane can be normalized by G_3 then we obtain

Theorem 1.2 *Let $G_3 = R^2 \backslash \{-2r_2, 0, 2r_1\}, r_1 > 0, r_2 > 0$, and $\widetilde{\gamma} \subset G_3$ be a closed rectifiable curve enclosing $\{-2r_2, 0, 2r_1\}$. Let $d(\widetilde{\gamma}, \{-2r_2, 0, 2r_1\})$ be the shortest Euclidean distance of $\widetilde{\gamma}$ to the boundary ∂G_3. Then*

$$\ell_{k_{G_3}}(\widetilde{\gamma}) \geq \left(\pi - \arctan \frac{y_1}{r_1}\right)\sqrt{1 + \frac{r_1^2}{y_1^2}} + \frac{5}{2}\pi - 2\arctan \frac{y}{r_2}, \tag{1.3}$$

where y_1 satisfies

$$d = \sqrt{r_1^2 + y_1^2}\, e^{\left(\arctan \frac{y_1}{r_1} - \pi\right)\frac{r_1}{y_1}}, \tag{1.4}$$

and $y = \max\{h_1(y_1), h_2(y_1)\}$ with

$$h_1(y_1) = \sqrt{y_1^2 + r_1^2}\, e^{\arctan \frac{3\pi}{4} - \frac{y_1}{r_1}}, \quad \sqrt{y_1^2 + r_1^2} \geq \sqrt{2} r_2 e^{\left(\arctan \frac{y_1}{r_1} - \frac{3\pi}{4}\right)};$$
$$h_2(y_1) = \frac{r_2\left(\frac{3\pi}{4} - \arctan \frac{y_1}{r_1}\right)}{\log r_2 - \log\sqrt{r_1^2 + y_1^2}}, \quad \sqrt{y_1^2 + r_1^2} < \sqrt{2} r_2 e^{\left(\arctan \frac{y_1}{r_1} - \frac{3\pi}{4}\right)}.$$

The estimate of (1.3) tends to 2π as $d \to \infty$ and it is the same as the estimate of (1.2) in G_2 as $r_2 \to 0$. The estimate of (1.3) in G_3 tends to 2π as $r_2 \to 0, r_1 \to 0$. By introducing the parameters of r_1 and r_2, we build bridges among G_3, G_2, G_1 and these estimates are asymptotically sharp.

If the thrice punctured plane can be normalized by G_3' then we have

Theorem 1.3 *Let $G_3' = R^2 \backslash \{r, re^{i2\alpha_1}, re^{i2(\alpha_1+\alpha_2)}\}, r > 0, \alpha_1 > 0, \alpha_2 > 0$ with the condition that $\alpha_1 + \alpha_2 < \pi$ and $\widetilde{\gamma} \subset G_3'$ be a closed rectifiable curve which encloses $\{r, re^{i2\alpha_1}, re^{i2(\alpha_1+\alpha_2)}\}$. Let $d(\widetilde{\gamma}, \{r, re^{i2\alpha_1}, re^{i2(\alpha_1+\alpha_2)}\})$ represent the shortest Euclidean distance of $\widetilde{\gamma}$ to the boundary $\partial G_3'$. Then*

$$\ell_{k_{G_3'}}(\widetilde{\gamma}) \geq \min\{f_1(x_1), f_2(x_1)\} + 2\pi - \alpha_1, \tag{1.5}$$

where $\hat{k}_1 = \tan(\min\{\alpha_1, \pi - \alpha_1 - \alpha_2\})$,

$$f_1(x_1) = \left(\pi - \arctan\frac{\hat{k}_1}{r - x_1}\right)\sqrt{1 + \frac{\hat{k}_1^2 r^2}{\left[\left(1 + \hat{k}_1^2\right)x_1 - r\right]^2}}, \quad \frac{r}{1 + \hat{k}_1^2} < x_1 \leq r,$$

and x_1 satisfies

$$\log\frac{\sqrt{(r - x_1)^2 + \hat{k}_1^2 x_1^2}}{d} = \frac{\hat{k}_1 r}{\left(1 + \hat{k}_1^2\right)x_1 - r}\left(\pi - \arctan\frac{\hat{k}_1 x_1}{r - x_1}\right).$$

Furthermore

$$f_2(x_1) = \arctan\frac{\hat{k}_1 x_1}{x_1 - r}\sqrt{1 + \frac{\hat{k}_1^2 r^2}{\left[\left(1 + \hat{k}_1^2\right)x_1 - r\right]^2}}, \quad x_1 > r,$$

and x_1 satisfies

$$\log\frac{\sqrt{(r - x_1)^2 + \hat{k}_1^2 x_1^2}}{d} = \frac{\hat{k}_1 r}{\left(1 + \hat{k}_1^2\right)x_1 - r}\arctan\frac{\hat{k}_1 x_1}{x_1 - r}.$$

The estimate of (1.5) tends to 2π as $r \to 0$. This means our estimate in G_3' can reduce to the lower bound in G_1 as $r \to 0$ and our estimate of (1.5) is asymptotically sharp, too.

It is known that the angle sum of a Euclidean triangle is equal to π and the angle sum of a hyperbolic triangle is less than π [2]. Moreover, Klén [9] proved that the angle sums of a quasihyperbolic triangle and a quasihyperbolic trigon in $G_1 = R^2 \backslash \{o\}$ are π and 3π, respectively. We would like to ask whether the angle sums of quasihyperbolic triangle and trigon in multiply punctured domainsare the same as those in the once punctured domain. We give an example of the angle sum of a quasihyperbolic trigon in G_3 is equal to 2π and prove that the cosine inequality does not hold in twice or thrice punctured planes.

2 Preliminary Knowledge and Lemmas

If the geodesic $\gamma : z_1 \curvearrowright z_2 \subset G_1$ and $\angle(z_1, o, z_2) = \alpha \in [0, \pi]$, then it is a subset of a logarithmic spiral. The polar equation of a logarithmic spiral determined by two distinct points $z_1, z_2 \in G_1$ is given by

$$r = ae^{b\theta}, \quad a = |z_1|e^{-b \arg z_1}, \quad b = \frac{1}{\alpha} \log \frac{|z_2|}{|z_1|}, \tag{2.1}$$

where $\alpha = \angle(z_1, o, z_2)$. There are many interesting properties of the logarithmic spiral. We define a ray by $\iota(z) = \{z : z = \lambda z_1, \lambda \in (0, \infty), z_1 \in G_1\}$. The angle between $\iota(z)$ and the tangent of the logarithmic spiral at an intersection point is given by

$$\arctan \frac{1}{b} = \arctan \frac{\alpha}{\log |z_2| - \log |z_1|}. \tag{2.2}$$

From the above formula we can deduce that the angle dose not depend on z and it is a constant. In the case $b = \infty$ the logarithmic spiral degenerates to a ray and in the case $b = 0$ the logarithmic spiral degenerates to a circle.

For $z_1, z_2 \in G_1$, then a geodesic $\gamma : z_1 \curvearrowright z_2$ is uniquely determined by (1.1) if $\angle(z_1, o, z_2) < \pi$ (see [9]). If $\angle(z_1, o, z_2) = \pi$, then there exist two symmetric geodesics connecting z_1 and z_2. Moreover, the quasihyperbolic length of each geodesic satisfies $k_{G_1}(\gamma) \geq \pi$ and the equality occurs if $|z_1| = |z_2|$. If $\gamma \in G_1$ is a closed rectifiable curve enclosing $\{o\}$, then $k_{G_1}(\gamma) \geq 2\pi$.

The quasihyperbolic distance and the quasihyperbolic geodesic between a point and a line ℓ are defined as follows.

Definition 2.1 The quasihyperbolic distance from a point p to a line ℓ in a proper domain G is defined by

$$\widetilde{k}_G(p, \ell) = \inf_z \{k_G(p, z), z \in \ell\} \tag{2.3}$$

and $\widetilde{\gamma} : p \curvearrowright z$ is the quasihyperbolic geodesic between the point p and the line ℓ, where z is a point reaching the inf value in (2.3).

We note that for arbitrary $z_1, z_2 \in R^2\backslash\{z_0'\}, z_0'(x_0'', y_0'') \in R^2$, the quasihyperbolic length of the geodesic $\gamma : z_1 \curvearrowright z_2$ is invariant under an inversion, stretching and rotation with respect to z_0'. To study the quasihyperbolic distance from a point $p'(x_0', y_0')$ to a line $\ell' = \{(x, y) : y = \hat{k}'x + c'\}, z_0'\overline{\in}\ell'$ in $R^2\backslash\{z_0'\}$ is equivalent to study the quasihyperbolic distance from a fixed point $p(x_0, 0), x_0 = |p' - z_0'|$ to line $\ell = \{(x, y) : y = \hat{k}x + c, c \neq 0\}$ in $G_1 = R^2\backslash\{o\}$, where $\hat{k}, c$ depend on $\hat{k}', z_0'$ and c. In fact, we have

Lemma 2.2 *Let $p(x_0, 0)$, $x_0 > 0$, be a point in $G_1 = R^2\backslash\{o\}$ and a line $\ell = \{(x, y) : y = \hat{k}x + c, c \neq 0\}$, then the quasihyperbolic distance from the point p to the line ℓ is given by*

$$\widetilde{k}_{G_1}(p, \ell) = |\arg z_0|\sqrt{1+b^2} = |\arg z_0|\sqrt{1 + \frac{c^2}{\left[\left(1+\hat{k}^2\right)u_0 + \hat{k}c\right]^2}}, \tag{2.4}$$

where the point $z_0(u_0, v_0) \in \ell$ satisfies

$$|c \arg z_0| = \left(\left(1+\hat{k}^2\right)u_0 + \hat{k}c\right)\left|\log\frac{\sqrt{u_0^2 + \left(\hat{k}u_0 + c\right)^2}}{x_0}\right|. \tag{2.5}$$

Moreover, the geodesic (logarithmic spiral) $\gamma : p \curvearrowright z_0$ is orthogonal to the line ℓ.

Proof Let $z(u, v)$ be a point on the line $\ell = \{(x, y) : y = \hat{k}x + c, c \neq 0\}$. We divide ℓ into six cases with respect to two coefficients $\hat{k}$ and c.

Case 1. Let $-\infty < \hat{k} \leq 0$, $c > 0$. If $x_0 = -\frac{c}{\hat{k}}$, then p is just a point on the line ℓ and it is trival. Assume $0 < x_0 < -\frac{c}{\hat{k}}$, we have $\arg z = \arctan(\hat{k} + \frac{c}{u}) > 0$ (see Fig. 1a). Let $f(u)$ be equal to the square of the quasihyperbolic length between z and p, then

$$f(u) = k^2_{G_1}(z, p) = \arctan^2\left(\hat{k} + \frac{c}{u}\right) + \log^2\frac{\sqrt{u^2 + \left(\hat{k}u + c\right)^2}}{x_0}. \tag{2.6}$$

By some straightforward computations, we conclude that $-\frac{\hat{k}c}{1+\hat{k}^2} < u < -\frac{c}{\hat{k}}$. Then

$$f'(u) = \frac{2\left(\left(1+\hat{k}^2\right)u + \hat{k}c\right)}{u^2 + \left(\hat{k}u + c\right)^2}\left[-\frac{c\arctan\left(\hat{k} + \frac{c}{u}\right)}{\left(1+\hat{k}^2\right)u + \hat{k}c} + \log\frac{\sqrt{u^2 + \left(\hat{k}u + c\right)^2}}{x_0}\right]. \tag{2.7}$$

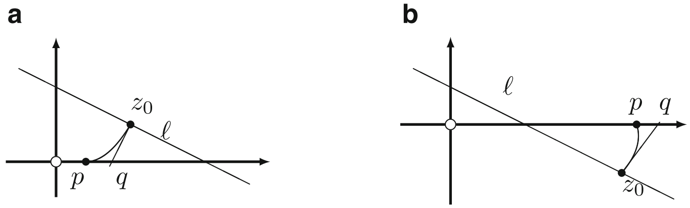

Fig. 1 **a** The case of negative straight slop and positive intersection point has a foot point of positive imagine part. **b** The case of negative straight slop and positive intersection point has a foot point of negative imagine part

Next we prove the uniqueness of the point z. Let

$$g(u) = \log \frac{\sqrt{u^2 + \left(\hat{k}u + c\right)^2}}{x_0} - \frac{c \arctan\left(\hat{k} + \frac{c}{u}\right)}{\left(1 + \hat{k}^2\right) u + \hat{k}c},$$

then we get

$$g'(u) = \frac{1 + \hat{k}^2}{\left(1 + \hat{k}^2\right) u + \hat{k}c} \left[1 + \frac{c \arctan\left(\hat{k} + \frac{c}{u}\right)}{\left(1 + \hat{k}^2\right) u + \hat{k}c} \right].$$

According to the relation $-\frac{\hat{k}c}{\hat{k}^2+1} < u < -\frac{c}{\hat{k}}$, we have $(1 + \hat{k}^2)u + \hat{k}c > 0$. So, we get $g'(u) > 0$. Furthermore, $g(u) \to -\infty$ as $u \to -\frac{\hat{k}c}{\hat{k}^2+1}$ and $g(u) \to \log \frac{|c|}{x_0|\hat{k}|} > 0$ as $u \to -\frac{c}{\hat{k}}$. Then $g(u) = 0$ and $f'(u) = 0$ have a common root u_0. Moreover, u_0 is unique and $f''(u_0) > 0$. Then $f(u)$ takes the minimum value at u_0. Because $f'(u_0) = 0$, we obtain (2.5). Combining (2.5) with (2.6) we obtain (2.4).

Let φ_1 be the included angle between the ray oz_0 and the tangent line qz_0, where q is the intersection point of the tangent line at z_0 and the real axis. By (2.1) and (2.2) we have

$$\varphi_1 = \arctan \frac{1}{b} = \arctan \frac{2 \arctan\left(\hat{k} + \frac{c}{u_0}\right)}{\log \frac{u_0^2 + \left(\hat{k}u_0 + c\right)^2}{x_0^2}}.$$

Using the relation (2.5) we get

$$\varphi_1 = \arctan \frac{\left(1 + \hat{k}^2\right) u_0 + \hat{k}c}{c}.$$

Let φ_2 be the included angle of the ray oz_0 and ℓ, we conclude that

$$\varphi_2 = \arctan\left(\hat{k} + \frac{c}{u_0}\right) - \arctan \hat{k}.$$

Then we get

$$\tan \varphi_2 = \frac{\tan\left(\arctan\left(\hat{k} + \frac{c}{u_0}\right)\right) - \tan\left(\arctan \hat{k}\right)}{1 + \tan\left(\arctan\left(\hat{k} + \frac{c}{u_0}\right)\right) \tan\left(\arctan \hat{k}\right)} = \frac{c}{\left(1 + \hat{k}^2\right) u_0 + \hat{k}c}.$$

By the equality $\varphi_2 = \frac{\pi}{2} - \arctan \frac{1}{\tan \varphi_2}$, $0 < \varphi_2 \le \frac{\pi}{2}$, we have another expression of φ_2:

$$\varphi_2 = \frac{\pi}{2} - \arctan \frac{\left(1+\hat{k}^2\right) u_0 + \hat{k}c}{c} = \frac{\pi}{2} - \arctan\left(\frac{1+\hat{k}^2}{c} u_0 + \hat{k}\right).$$

So the included angle of the line ℓ and the tangent line qz_0 is

$$\varphi_1 + \varphi_2 = \arctan \frac{\left(1+\hat{k}^2\right) u_0 + \hat{k}c}{c} + \frac{\pi}{2} - \arctan\left(\frac{1+\hat{k}^2}{c} u_0 + \hat{k}\right) = \frac{\pi}{2}.$$

Thus the shortest logarithmic spiral $\gamma : p \curvearrowright z_0$ of p to ℓ is orthogonal to the line ℓ at z_0.

When the point $p(x_0, 0)$ satisfies $x_0 > -\frac{c}{\hat{k}}$, then $\arctan(\hat{k} + \frac{c}{u}) < 0$ (see Fig. 1b). We define $f(u)$ by

$$f(u) = k_{G_1}^2(z, p) = \arctan^2\left(\hat{k} + \frac{c}{u}\right) + \log^2 \frac{x_0}{\sqrt{u^2 + \left(\hat{k}u + c\right)^2}}. \tag{2.8}$$

If z_3 satisfies $\arg z_3 = \arg z$, $z_3 \in \ell$, $\Re z_3 < -\frac{c}{\hat{k}}$, then $|z_3| < |z|$ and $k_{G_1}(z_3, p) > k_{G_1}(z, p)$. So we conclude that $u > -\frac{c}{\hat{k}}$. Thus we get

$$f'(u) = -\frac{2\left(\left(1+\hat{k}^2\right) u + \hat{k}c\right)}{u^2 + \left(\hat{k}u + c\right)^2} \left[\frac{c \arctan\left(\hat{k} + \frac{c}{u}\right)}{\left(1+\hat{k}^2\right) u + \hat{k}c} + \log \frac{x_0}{\sqrt{u^2 + \left(\hat{k}u + c\right)^2}}\right]. \tag{2.9}$$

Next we will prove the uniqueness of the point z. Let

$$g(u) = \log \frac{x_0}{\sqrt{u^2 + (\hat{k}u + c)^2}} + \frac{c \arctan\left(\hat{k} + \frac{c}{u}\right)}{\left(1+\hat{k}^2\right) u + \hat{k}c}.$$

Then

$$g'(u) = -\frac{\left(1+\hat{k}^2\right)}{\left(1+\hat{k}^2\right) u + \hat{k}c} \left[1 + \frac{c}{\left(1+\hat{k}^2\right) u + \hat{k}c} \arctan\left(\hat{k} + \frac{c}{u}\right)\right].$$

Since $u > -\frac{c}{\hat{k}} > 0$, the inequality $(1+\hat{k}^2)u + \hat{k}c > -\frac{c}{\hat{k}} > 0$ holds. Let

$$h(u) = \arctan\left(\hat{k} + \frac{c}{u}\right) + u\frac{1+\hat{k}^2}{c} + \hat{k}.$$

Then

$$h'(u) = -\frac{c}{u^2 + (\hat{k}u + c)^2} + \frac{1 + \hat{k}^2}{c} > -\frac{\hat{k}^2}{c} + \frac{1}{c} + \frac{\hat{k}^2}{c} > 0.$$

So $h(u)$ is an increasing function. Hence, we have $h(u) > h(-\frac{c}{\hat{k}}) = -\frac{1}{\hat{k}} > 0$ and $g'(u) < 0$. Moreover, $g(u) < 0$ as $u \to +\infty$ and $g(u) \to \log \frac{x_0|\hat{k}|}{|c|} > 0$ as $u \to -\frac{c}{\hat{k}}$. Thus the equation $g(u) = 0$ and $f'(u) = 0$ have a common root u_0. By concrete computation, we have that u_0 is unique and the inequality $f''(u_0) > 0$ holds. So $f(u)$ takes the minimum value at u_0. We get (2.5) by $f'(u_0) = 0$. By (2.5) and (2.8), we obtain (2.4). Using the same method as before, the logarithmic spiral $\gamma : p \curvearrowright z_0$ is also orthogonal to the line ℓ.

Let $0 < \hat{k} < +\infty, c < 0$ be Case 2 (see Fig. 2), $0 \leq \hat{k} < +\infty, c > 0$ Case 3 (see Fig. 3) and $-\infty < \hat{k} \leq 0, c < 0$ Case 4 (see Fig. 4). By symmetry, one can use the method of Case 1 to deduce the proofs of these three cases. For simplicity, we omit these proofs.

Case 5. Let $\hat{k} = \infty, c > 0$, the straight line ℓ can be expressed by $x = c$ (see Fig. 5). Then the point $z \in \ell$ is just the point $q(c, 0)$ and geodesic line $\gamma[p, q]$ is orthogonal to the line $x = c$. The quasihyperbolic distance from the point $p(x_0, 0)$ to ℓ is $\widetilde{k}_{G_1}(p, \ell) = |\log \frac{c}{x_0}|$.

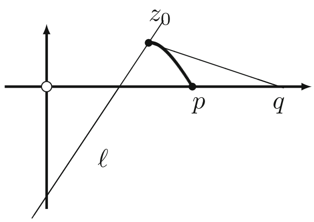

Fig. 2 The case that the straight slope and the intersection point are positive

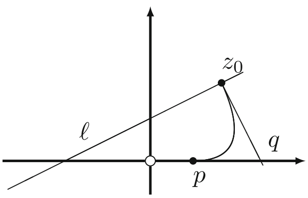

Fig. 3 The case that the straight slop is positive but the intersection point is negative

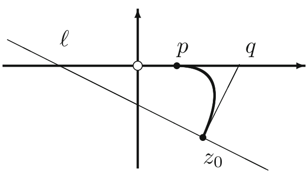

Fig. 4 The case that the straight slop and the intersection point are both negative

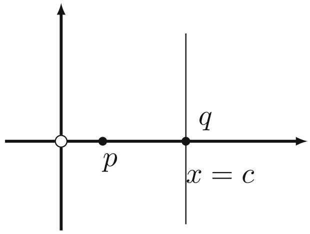

Fig. 5 The vertical case of a positive intersection point

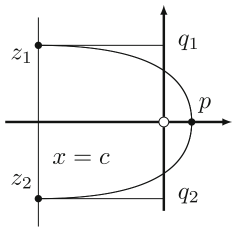

Fig. 6 The vertical case of a negative intersection point

Case 6. If $\hat{k} = \infty, c < 0$, then ℓ is $x = c$ (see Fig. 6). By the symmetry, there are two points $z_1(u, 0), z_2(-u, 0) \in \ell, u > 0$ satisfying that $\gamma : z_1 \curvearrowright p, \gamma : z_2 \curvearrowright p$ are both geodesics between p and ℓ. Moreover, u satisfies

$$\frac{c}{u}\left(\pi - \arctan\frac{u}{c}\right) = \log\frac{\sqrt{u^2 + c^2}}{d}.$$

The above equation has a unique root u_0 and $\gamma : z_1 \curvearrowright p, \gamma : z_2 \curvearrowright p$ are orthogonal to the line $x = c$. □

Remark 2.3 If $c = 0$, then the line ℓ degenerates into two rays $y = \hat{k}x(x > 0)$ and $y = \hat{k}x(x < 0)$. The quasihyperbolic distance from the point $p(x_0, 0), x_0 > 0$ to the ray $y = \hat{k}x, x > 0$ (or $y = \hat{k}x, x < 0$) is $\min\{|\arctan\hat{k}|, \pi - |\arctan\hat{k}|\}$ and the geodesic is geodesic arc. Especially, if $\hat{k} = 0$, then ℓ reduces to the positively real axis and the negatively real axis. The quasihyperbolic distance from the point $p(x_0, 0), x_0 > 0$ to the negatively real axis is π and the geodesics are the upper half circle and the lower half circle. The case of positively real axis is trivial.

An arbitrary twice punctured plane can be normalized by $G_2 = R^2 \setminus \{-r, r\}, r > 0$. We generalize Lemma 5.2 in [9] as follows.

Lemma 2.4 *Let* $z_1 \in G_2 = R^2 \setminus \{-r, r\}, r > 0, 0 < \arg z_1 < \varphi, F = \{z \in R^2 : \arg z = \varphi\}, 0 < \varphi \leq \frac{\pi}{2}$, *and* $E = \{z \in R^2 : \arg z = 0, \Re z > r\}$. *Then*

$$\widetilde{k}_{G_2}(E, z_1) + \widetilde{k}_{G_2}(F, z_1) \geq \widetilde{k}_{G_2}(F, z_2),$$

where $z_2 = r + d(z_1)$, *and* $d(z_1) = d(z_1, \partial G_2)$ *represents the Euclidean distance from* z_1 *to the boundary* ∂G_2.

Proof According to Lemma 2.2, the following inequality

$$\widetilde{k}_{G_2}(F, z_2) \le k_{G_2}(z_2, z_3) \le k_{G_2}(z_2, z_1) + k_{G_2}(z_3, z_1)$$

holds, where $z_2 \in E$, $z_3 \in F$ satisfy

$$k_{G_2}(z_1, z_2) = \widetilde{k}_{G_2}(z_1, E), \quad k_{G_2}(z_1, z_3) = \widetilde{k}_{G_2}(z_1, F).$$

So this lemma is proved. □

Lemma 2.5 *Denote by* $F = \{z \in R^2 : \arg z = \varphi\}, 0 < \varphi \le \frac{\pi}{2}$, *and* $E = \{z \in R^2 : \arg z = 0, \Re z > r\}$, *respectively. Let* z_1 *be a point in* $G_2 = R^2 \backslash \{-r, r\}$ *with* $r > 0$ *and* $0 < \arg z_1 < \varphi$. *If* $p_1(d_1 + r, 0) \in E$, $p_2(d_2, \varphi) \in F$ *are two points satisfy that* $d_2 \cos\varphi > r$ *and* $0 < \varphi \le \frac{\pi}{2}$, *then*

$$k_{G_2}(p_1, p_2) \ge \varphi, \tag{2.10}$$

and $k_{G_2}(p_1, p_2) \to \varphi$ *as* $d_1 \to \infty$, $\frac{d_2}{d_1} \to 1$.

Proof According to (2.1), the quasihyperbolic length from p_1 to p_2 can be expressed by

$$k_{G_2}(p_1, p_2) = \sqrt{\arctan^2\left(\frac{d_2 \sin\varphi}{d_2 \cos\varphi - r}\right) + \log^2 \frac{\sqrt{(d_2 \cos\varphi - r)^2 + d_2^2 \sin^2\varphi}}{d_1}}.$$

It is obvious to get (2.10). By Lemma 2.2 and (2.5) we have

$$e^{-\frac{\pi \cot\varphi}{2}} \sqrt{d_2^2 - 2rd_2 \cos\varphi + r^2} < d_1 < \sqrt{d_2^2 + r^2}.$$

Moreover, $k_{G_2}(p_1, p_2) \to \varphi$ as $d_1 \to \infty$ and $\frac{d_2}{d_1} \to 1$, thus the lower bound φ is asymptotically sharp. □

3 Proofs of the Main Theorems

We generalize Theorem B to the planar domain $G_2 = R^2 \backslash \{-r, r\}$. Let $\widetilde{\gamma} \subset G_2$ be a rectifiable curve enclosing $\{-r, r\}$. We can find a lower bound for $\ell_{k_{G_2}}(\widetilde{\gamma})$.

Proof of Theorem 1.1 By Lemma 2.4, we can assume that $p(r + d, 0)$ belongs to $\widetilde{\gamma}$ and $d = d(\widetilde{\gamma}, \{-r, r\})$ represents the shortest Euclidean distance of $\widetilde{\gamma}$ to the boundary ∂G_2. By Lemma 2.5, we have the quasihyperbolic lengths of the subarcs of $\widetilde{\gamma}$ in the second, third and fourth quadrant have a common lower bound of $\pi/2$. Let $\ell = \{z \in R^2 : \Re z = 0, \Im z > 0\}$ and $z = yi \in \ell$, $y > 0$. Therefore we have

$$\ell_{k_{G_2}}(\widetilde{\gamma}) \ge k_{G_2}(\widetilde{\gamma}) \ge \widetilde{k}_{G_2}(p, \ell) + \frac{3}{2}\pi. \tag{3.1}$$

By Lemma 2.2, we have

$$\widetilde{k}_{G_2}(p,\ell)=\sqrt{\left(\pi-\arctan\frac{y}{r}\right)^2+\log^2\frac{\sqrt{r^2+y^2}}{d}},\tag{3.2}$$

where y satisfies

$$\log\frac{\sqrt{y^2+r^2}}{d}=\frac{r}{y}\left(\pi-\arctan\frac{y}{r}\right).\tag{3.3}$$

By substituting (3.3) into (3.2) and we get (1.2) from (3.1).

By (3.3), we have

$$d=h(y)=\sqrt{y^2+r^2}e^{\frac{r}{y}(\arctan\frac{y}{r}-\pi)},$$

and

$$h'(y)=\sqrt{y^2+r^2}e^{\frac{r}{y}(\arctan\frac{y}{r}-\pi)}\left(\frac{1}{y}+\frac{r\left(\pi-\arctan\frac{y}{r}\right)}{y^2}\right)>0.$$

So $d(\widetilde{\gamma},\{-r,r\})$ increases in y for a fixed r. Hence, Theorem 1.1 is proved. □

We can normalize $R^2\backslash\{z_1,z_2,z_3\}$ by $G_3=R^2\backslash\{-2r_2,o,2r_1\}, r_1>0, r_2>0$ when z_1,z_2,z_3 lie in a straight line. Next, we will give our estimate about the lower bound for quasihyperbolic length of $\widetilde{\gamma}$ in G_3.

Proof of Theorem 1.2 According to Lemma 2.2, we assume that $p(r+d,0)\in\widetilde{\gamma}$ and $d=d(\widetilde{\gamma},\{-2r_2,o,2r_1\})$ represents the shortest Euclidean distance of $\widetilde{\gamma}$ to the boundary ∂G_3. Let $\widetilde{\ell_1}=\{(x,y):x=r_1,y>0\}$, $\widetilde{\ell_2}=\{(x,y):x=-r_2,y>0\}$, $\widetilde{\ell_3}=\{(x,y):y=0,x<0\}$. Gehring and Osgood [4] proved that there always exists a quasihyperbolic geodesic γ' connecting z_1 and z_2. Then there is at least a point $p_2\in\widetilde{\ell_2}$ satisfying the quasihyperbolic geodesic $\gamma:p_2\curvearrowright\widetilde{\ell_3}$ is a subarc of the quasihyperbolic geodesic $\gamma:p\curvearrowright\widetilde{\ell_3}$. Moreover, there exists a point $p_3\in\widetilde{\ell_3}$ satisfying $\gamma:p_2\curvearrowright p_3=\gamma:p_2\curvearrowright\widetilde{\ell_3}$ and depending on p_2 by Lemma 2.4. There is also at least one point $p_1\in\widetilde{\ell_1}$ satisfying $\gamma:p\curvearrowright\widetilde{\ell_3}=\gamma:p\curvearrowright p_1\cup\gamma:p_1\curvearrowright p_2\cup\gamma:p_2\curvearrowright p_3$. Let $p_1(r_1,y_1)\in\widetilde{\ell_1}$, $p_2(-r_2,y_2)\in\widetilde{\ell_2}$, $p_3(-2r_2-\sqrt{r_2^2+y_2^2},0)\in\widetilde{\ell_3}$. By symmetry, we get $\ell_{k_{G_3}}(\widetilde{\gamma})\geq k_{G_3}(\widetilde{\gamma})\geq 2\widetilde{k}_{G_3}(p,\widetilde{\ell_3})$ and

$$\begin{aligned}2k_{G_3}\left(p,\widetilde{\ell_3}\right)&=\min_{y_1,y_2}2\left[k_{G_3}(p,p_1)+k_{G_3}(p_1,p_2)+k_{G_3}(p_2,p_3)\right]\\&=2\min_{y_1,y_2}\left[\sqrt{\left(\pi-\arctan\frac{y_1}{r_1}\right)^2+\log^2\frac{\sqrt{r_1^2+y_1^2}}{d}}\right.\\&\quad\left.+\sqrt{\left(\pi-\arctan\frac{y_1}{r_1}-\arctan\frac{y_2}{r_2}\right)^2+\frac{1}{4}\log^2\frac{r_2^2+y_2^2}{r_1^2+y_1^2}}+\pi-\arctan\frac{y_2}{r_2}\right].\end{aligned}$$

By Lemma 2.2, we have

$$\sqrt{\left(\pi - \arctan\frac{y_1}{r_1}\right)^2 + \log^2\frac{\sqrt{r_1^2+y_1^2}}{d}} \geq \left(\pi - \arctan\frac{y_1}{r_1}\right)\sqrt{1+\frac{r_1^2}{y_1^2}},$$

and the equality holds when y_1 satisfies

$$d = \sqrt{r_1^2+y_1^2}\,e^{\left(\arctan\frac{y_1}{r_1}-\pi\right)\frac{r_1}{y_1}}. \tag{3.4}$$

Next we assume that the point $p_2'(r_2, y)$ satisfies $y > 0$. Let

$$g(y) = k_{G_3}^2(p_1, p_2') = \left(\pi - \arctan\frac{y_1}{r_1} - \arctan\frac{y}{r_2}\right)^2 + \log^2\frac{\sqrt{r_2^2+y^2}}{\sqrt{r_1^2+y_1^2}}.$$

By Lemma 2.2, $g(y)$ takes the minimum at $p_2'(r_2, y)$ and y is the solution of the equation

$$\sqrt{y_1^2+r_1^2} = \sqrt{r_2^2+y^2}\,e^{\frac{r_2}{y}\left(\arctan\frac{y_1}{r_1}+\arctan\frac{y}{r_2}-\pi\right)}. \tag{3.5}$$

We assume that $y \geq r_2$. Then $\sqrt{y_1^2+r_1^2} \geq \sqrt{2}r_2e^{(\arctan\frac{y_1}{r_1}-\frac{3\pi}{4})}$. By (3.5), we have

$$\sqrt{y_1^2+r_1^2} \geq ye^{\frac{r_2}{y}\left(\arctan\frac{y_1}{r_1}+\arctan\frac{y}{r_2}-\pi\right)} \geq ye^{\left(\arctan\frac{y_1}{r_1}-\frac{3\pi}{4}\right)},$$

and hence $y \leq \sqrt{y_1^2+r_1^2}\,e^{(\frac{3\pi}{4}-\arctan\frac{y_1}{r_1})} = h_1(y_1)$. Therefore

$$k_{G_3}(p_1, p_2) + k_{G_3}(p_2, p_3) \geq \pi - \arctan\frac{h_1(y_1)}{r_2}, \tag{3.6}$$

where p_1, p_2, p_3 satisfy $\gamma : p \curvearrowright \widetilde{\ell}_3 = \gamma : p \curvearrowright p_1 \cup \gamma : p_1 \curvearrowright p_2 \cup \gamma : p_2 \curvearrowright p_3$.

If $0 < y < r_2$, then $\sqrt{y_1^2+r_1^2} < \sqrt{2}r_2e^{(\arctan\frac{y_1}{r_1}-\frac{3\pi}{4})} < r_2$. By (3.5), we get

$$\sqrt{y_1^2+r_1^2} > r_2e^{\frac{r_2}{y}\left(\arctan\frac{y_1}{r_1}-\frac{3\pi}{4}\right)},$$

and then $y < \frac{r_2(\frac{3\pi}{4}-\arctan\frac{y_1}{r_1})}{\log r_2-\log\sqrt{r_1^2+y_1^2}} = h_2(y_1)$. Hence we have

$$k_{G_3}(p_1, p_2) + k_{G_3}(p_2, p_3) \geq \pi - \arctan\frac{h_2(y_1)}{r_2}. \tag{3.7}$$

where p_1, p_2, p_3 satisfy $\gamma : p \curvearrowright \widetilde{\ell}_3 = \gamma : p \curvearrowright p_1 \cup \gamma : p_1 \curvearrowright p_2 \cup \gamma : p_2 \curvearrowright p_3$.

By Lemma 2.5, we conclude that the quasihyperbolic length of $\widetilde{\gamma}$ in the domain $\Omega = \{z = u + iv \subset R^2, u \geq r_1, v \leq 0\}$ has a lower bound $\frac{\pi}{2}$. Combining (3.6) with (3.7), we obtain (1.3). The proof of Theorem 1.2 is complete. □

If z_1, z_2, z_3 are not in a line, then there always exists a circumscribed circle of the triangle $\Delta_{z_1 z_2 z_3}$. The domain $R^2 \backslash \{z_1, z_2, z_3\}$ can be normalized by $G_3' = R^2 \backslash \{r, re^{i2\alpha_1}, re^{i2(\alpha_1+\alpha_2)}\}, r > 0, \alpha_1 > 0, \alpha_2 > 0, \alpha_1 + \alpha_2 < \pi$.

Proof of Theorem 1.3 Without loss of generalization, we assume that the intersection point of $\widetilde{\gamma}$ and the ray oz_1 is $p(r+d, 0)$ and $d = d(\widetilde{\gamma}, \{r, re^{i2\alpha_1}, re^{i2(\alpha_1+\alpha_2)}\})$ represents the shortest Euclidean distance of $\widetilde{\gamma}$ to the boundary $\partial G_3'$. Let $\alpha_3 = \pi - \alpha_1 - \alpha_2$ and $\alpha_1 = \min\{\alpha_1, \pi - \alpha_1 - \alpha_2\}$, then $0 < \alpha_1 < \frac{\pi}{2}$.

By Lemma 2.2, we assume that the shortest geodesic from the point p to the line $\ell = \{y = \hat{k}_1 x : \hat{k}_1 = \tan(\alpha_1)\}$ is γ_1. The point $p_1(x_1, \hat{k}_1 x_1)$ is the intersection point of γ_1 and ℓ. The point $p_1(x_1, \hat{k}_1 x_1)$ can be devided into two cases.

Case 1: If $\frac{r}{1+\hat{k}_1^2} < x_1 \leq r$, then

$$k_{G_3'}(\gamma_1) = \min_{x_1} \sqrt{\left(\pi - \arctan \frac{\hat{k}_1 x_1}{r - x_1}\right)^2 + \log^2 \frac{\sqrt{(r-x_1)^2 + \hat{k}_1^2 x_1^2}}{d}}.$$

Let $k_{G_3'}(\gamma_1) = f_1(x_1)$. Combining with Lemma 2.2, we have

$$f_1(x_1) = \left(\pi - \arctan \frac{\hat{k}_1 x_1}{r - x_1}\right) \sqrt{1 + \frac{\hat{k}_1^2}{\left[\left(1 + \hat{k}_1^2\right) x_1 - r\right]^2}},$$

where x_1 satisfies

$$\log \frac{\sqrt{(r-x_1)^2 + \hat{k}_1^2 x_1^2}}{d} = \frac{\hat{k}_1 r}{\left(1 + \hat{k}_1^2\right) x_1 - r} \left(\pi - \arctan \frac{\hat{k}_1 x_1}{r - x_1}\right).$$

Case 2: If $x_1 > r$, then

$$k_{G_3'}(\gamma_1) = \min_{x_1} \sqrt{\left(\arctan \frac{\hat{k}_1 x_1}{x_1 - r}\right)^2 + \log^2 \frac{\sqrt{(r-x_1)^2 + \hat{k}_1^2 x_1^2}}{d}}.$$

Let $k_{G_3'}(\gamma_1) = f_2(x_1)$. Combining with Lemma 2.2, we have

$$f_2(x_1) = \arctan \frac{\hat{k}_1 x_1}{x_1 - r} \sqrt{1 + \frac{\hat{k}_1^2 r^2}{\left[\left(1 + \hat{k}_1^2\right) x_1 - r\right]^2}},$$

where x_1 satisfies

$$\log \frac{\sqrt{(x_1 - r)^2 + \hat{k}_1^2 x_1^2}}{d} = \frac{\hat{k}_1 r}{\left(1 + \hat{k}_1^2\right) x_1 - r} \arctan \frac{\hat{k}_1 x_1}{x_1 - r}.$$

By Lemma 2.4, we know that the lower bound of the quasihyperbolic length of $\widetilde{\gamma}$ in domains of $\Omega_1 = \{z : \alpha_1 < \arg z < 2\alpha_1\}$, $\Omega_2 = \{z : 2\alpha_1 < \arg z < 2\alpha_1 + \alpha_2\}$, $\Omega_3 = \{z : 2\alpha_1 + \alpha_2 < \arg z < 2(\alpha_1 + \alpha_2)\}$, $\Omega_4 = \{z : 2(\alpha_1 + \alpha_2) < \arg z < 2(\alpha_1 + \alpha_2) + \alpha_3\}$, $\Omega_5 = \{z : 2(\alpha_1 + \alpha_2) + \alpha_3 < \arg z < 2\pi\}$ have lower bounds $\alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3$, respectively. Since $2(\alpha_1 + \alpha_2 + \alpha_3) = 2\pi$, we get

$$k_{G_3'}(\widetilde{\gamma}) \geq \min\{f_1(x_1), f_2(x_1)\} + 2\pi - \alpha_1.$$

The proof of Theorem 1.3 is complete. □

Klén [9] got the law of cosines and the inequality of cosines in the sense of quasihyperbolic metric in $R^2 \backslash \{o\}$. Huang, etc. [8] asserted that the inequality of cosines does not hold in $R^2 \backslash \{-1, 1\}$ (see Example 1 in [8]). The following example shows that the cosine law and the cosine inequality do not hold in a twice or thrice punctured plane.

Example Let $G_3 = R^2 \backslash \{-2r_2, 0, 2r_1\}$, $r_2 > r_1 > 0$, $z(r_1, y_1)$, $p(2r_1 + d, 0)$, $z_1(-d, 0)$, $z_2(r_1, -y_1)$, $z_3(r_1, \frac{1}{\sqrt{3}} r_1)$, $z_4(2r_1 + 4d, 0)$, $z_5(2r_1 + 4d \cos\beta, 4d \sin\beta)$, $0 < \beta < \frac{\pi}{2}$, $d > 0$ and $r_1 = y_1 = \frac{1}{\sqrt{2}} d e^{\frac{3\pi}{4}}$. Then

(1) $k_{G_3}^2(z_1, z_2) < k_{G_3}^2(z_1, p) + k_{G_3}^2(z_2, p) - 2k_{G_3}(z_1, p) k_{G_3}(z_2, p) \cos\angle(z_1, p, z_2)$;

(2) $k_{G_3}^2(z_3, p) > k_{G_3}^2(z_3, z) + k_{G_3}^2(z, p) - 2k_{G_3}(z_3, z) k_{G_3}(z, p) \cos\angle(p, z, z_3)$;

(3) $k_{G_3}^2(z_5, p) = k_{G_3}^2(z_4, p) + k_{G_3}^2(z_4, z_5) - 2k_{G_3}(z_4, p) k_{G_3}(z_4, z_5) \cos\angle(p, z_4, z_5)$.

Proof By Lemma 2.2, $\widetilde{\gamma} : p \curvearrowright z_1$ passes through z and is orthogonal to the ray $x = r_1$, $y > 0$. So we have

$$k_{G_3}(z_1, p) = 2k_{G_3}(z, p) = \frac{3\sqrt{2}\pi}{2}, \quad k_{G_3}(z_1, z_2) = k_{G_3}(z_2, p) = k_{G_3}(z, p) = \frac{3\sqrt{2}\pi}{4}.$$

According to (1.1), we get

$$b = \frac{4}{3\pi} \log \frac{\sqrt{r_1^2 + y_1^2}}{d} = 1, \quad \angle(z_1, p, z_2) = 2 \arctan \frac{1}{b} = \frac{\pi}{2}.$$

By the above equalities, we have the inequality (1) holds. Moreover, $\angle(z_1, p, z_2) + \angle(p, z_1, z_2) + \angle(z_1, z_2, p) = 2\pi$. This is different from the quasihyperbolic trigon in G_1 obtained by Klén [9].

By some straightforward calculations, we have

$$k_{G_3}(z_3, p) = \sqrt{\left(\pi - \arctan\frac{y_3}{r_1}\right)^2 + \log^2\frac{\sqrt{r_1^2 + y_3^2}}{d}} = \sqrt{(\frac{5\pi}{6})^2 + \left(\log\frac{\sqrt{2}}{\sqrt{3}} + \frac{3\pi}{4}\right)^2},$$

and

$$k_{G_3}(z_3, z) = \int_{\frac{r_1}{\sqrt{3}}}^{r_1} \frac{1}{\sqrt{r_1^2 + x^2}} dx = \ln\frac{1 + \sqrt{2}}{\sqrt{3}}, \quad \angle(p, z, z_3) = \frac{\pi}{2}.$$

Then the inequality (2) is proved.

Furthermore, we have

$$k_{G_3}(z_4, p) = \ln 4, \quad k_{G_3}(z_4, z_5) = \beta, \quad k_{G_3}(z_5, p) = \sqrt{\beta^2 + \ln^2 4}, \quad \angle(p, z_4, z_5) = \frac{\pi}{2}.$$

So we get the inequality (3). □

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